**February 1- 2 - 3, 2022, University of Namur**

Timoteo Carletti

UNIVERSITÉ DE NAMUR

Mattia Frasca

UNIVERSITÀ degli STUDI di CATANIA

# Complex Networks & Dynamical Systems

ECOLE DOCTORALE SCIENCES

COMPLEX

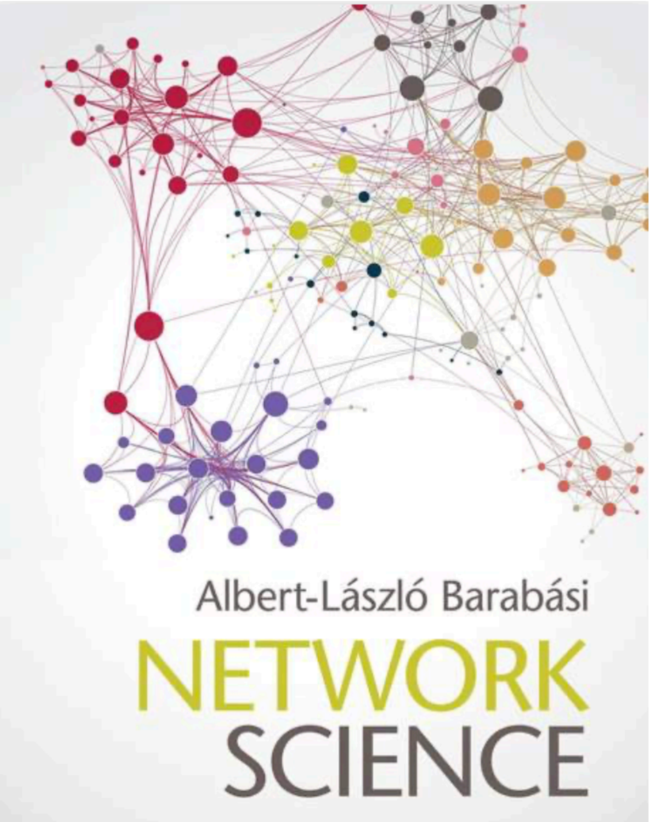
fnrs LA LIBERTÉ DE CHERCHER

Department of mathematics
UNamur

naxys Namur Institute for Complex Systems

UNIVERSITÉ DE NAMUR

# Motivation : We live in an interconnected world ...



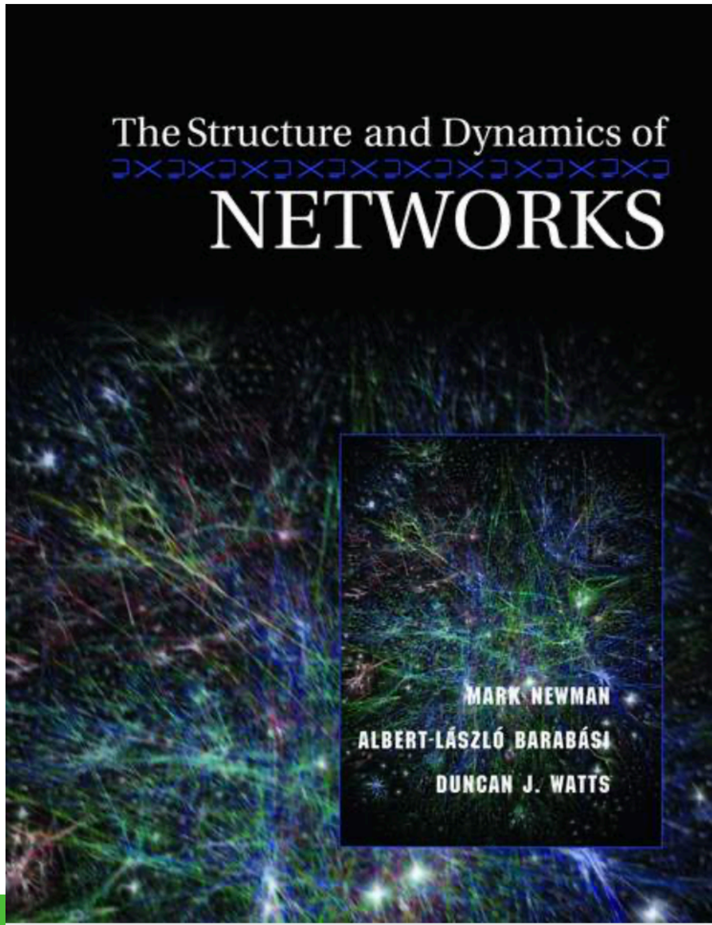
## Network Science A.-L. Barabási

Networks are everywhere, from the Internet, to social networks, and the genetic networks that determine our biological existence. Illustrated throughout in full colour, this pioneering textbook, spanning a wide range of topics from physics to computer science, engineering, economics and the social sciences, introduces network science to an interdisciplinary audience. From the origins of the six degrees of separation to explaining why networks are robust to random failures, the author explores how viruses like Ebola and H1N1 spread, and why it is that our friends have more friends than we do. Using numerous real-world examples, this innovatively designed text includes clear delineation between undergraduate and graduate level material. The mathematical formulas and derivations are included within Advanced Topics sections, enabling use at a range of levels. Extensive online resources, including films and software for network analysis, make this a multifaceted companion for anyone with an interest in network science.

## The Structure and Dynamics of Networks

A.-L. Barabási, M. Newman, D.J.Watts

From the Internet to networks of friendship, disease transmission, and even terrorism, the concept-and the reality-of networks has come to pervade modern society. But what exactly is a network? What different types of networks are there? Why are they interesting, and what can they tell us? In recent years, scientists from a range of fields-including mathematics, physics, computer science, sociology, and biology-have been pursuing these questions and building a new "science of networks." This book brings together for the first time a set of seminal articles representing research from across these disciplines. It is an ideal sourcebook for the key research in this fast-growing field. The book is organized into four sections, each preceded by an editors' introduction summarizing its contents and general theme. The first section sets the stage by discussing some of the historical antecedents of contemporary research in the area. From there the book moves to the empirical side of the science of networks before turning to the foundational modeling ideas that have been the focus of much subsequent activity. The book closes by taking the reader to the cutting edge of network science--the relationship between network structure and system dynamics. From network robustness to the spread of disease, this section offers a potpourri of topics on this rapidly expanding frontier of the new science.



# … where “basic” units interact each others

Support for the spread of

- Information
- Opinions
- Likes
- Viruses
- …

# … where "basic" units interact each others

At larger scale

support for the spread of goods

Support for the spread of

- Information
- Opinions
- Likes
- Viruses
- …

timoteo.carletti@unamur.be

# … where "basic" units interact each others

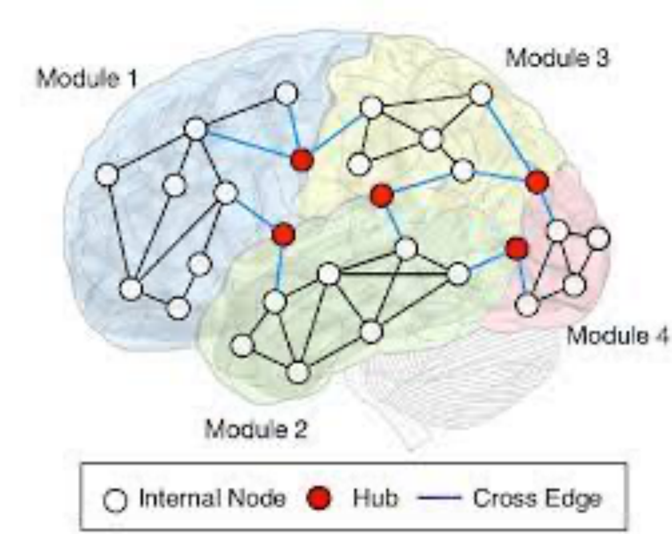
At larger scale

support for the spread of goods

Support for the spread of

- Information
- Opinions
- Likes
- Viruses
- …

At smaller scale



support for the spread of signals (memory, actions, thoughts, …)

# Research question: understand the system behaviour

Reductionist approach
(e.g., grand unified theory)

vs.

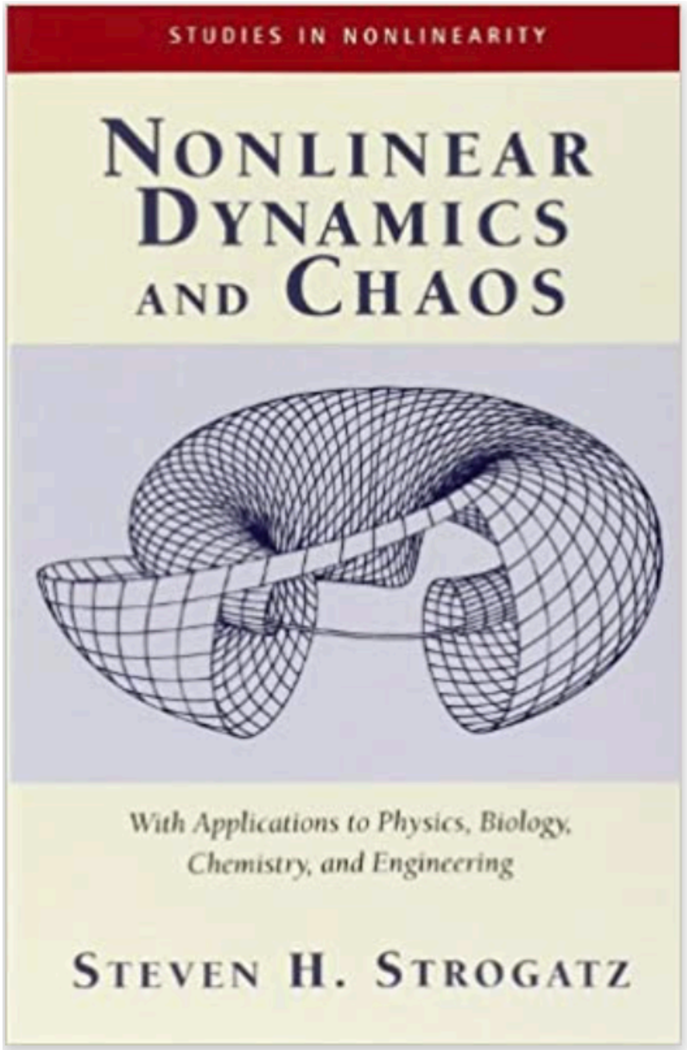
Holistic approach
(e.g., complex systems)

# Outlook of the lectures

- Some basic facts about dynamical systems theory;
- A short introduction about network theory;
- Networked dynamical systems: Turing patterns;
- Beyond network theory (if time will allow).

- Synchronisation on complex networks;
- Synchronisation on temporal networks;
- Network-based control of multi-agent systems.

# Dynamical Systems

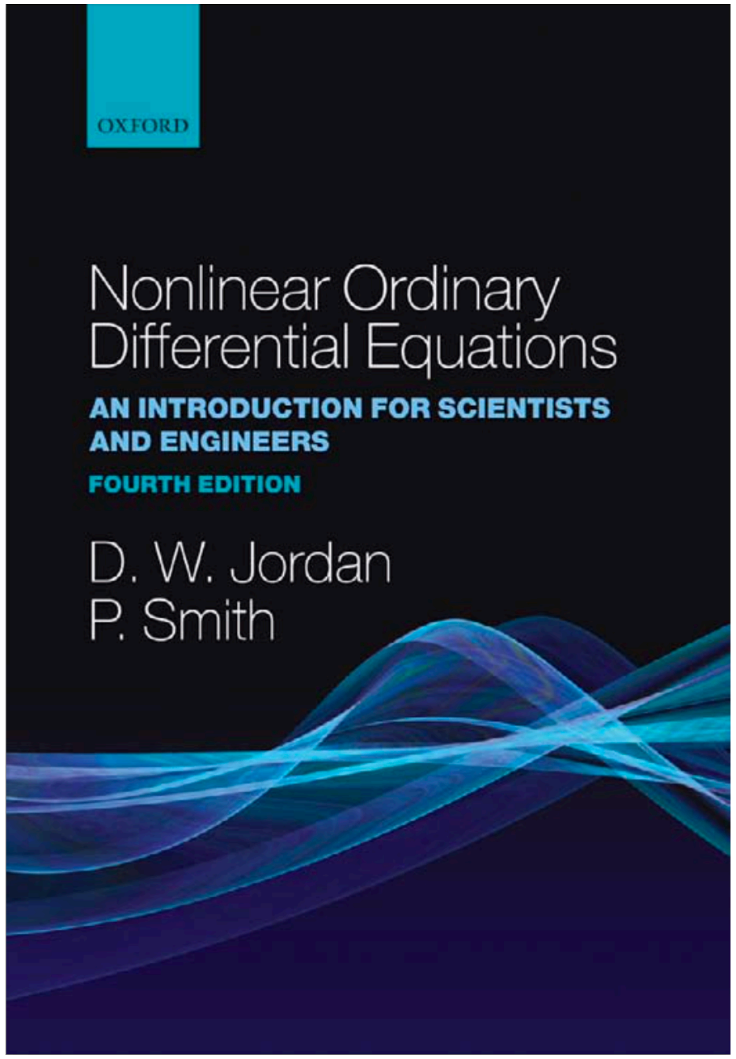


STUDIES IN NONLINEARITY

NONLINEAR DYNAMICS AND CHAOS

*With Applications to Physics, Biology, Chemistry, and Engineering*

STEVEN H. STROGATZ



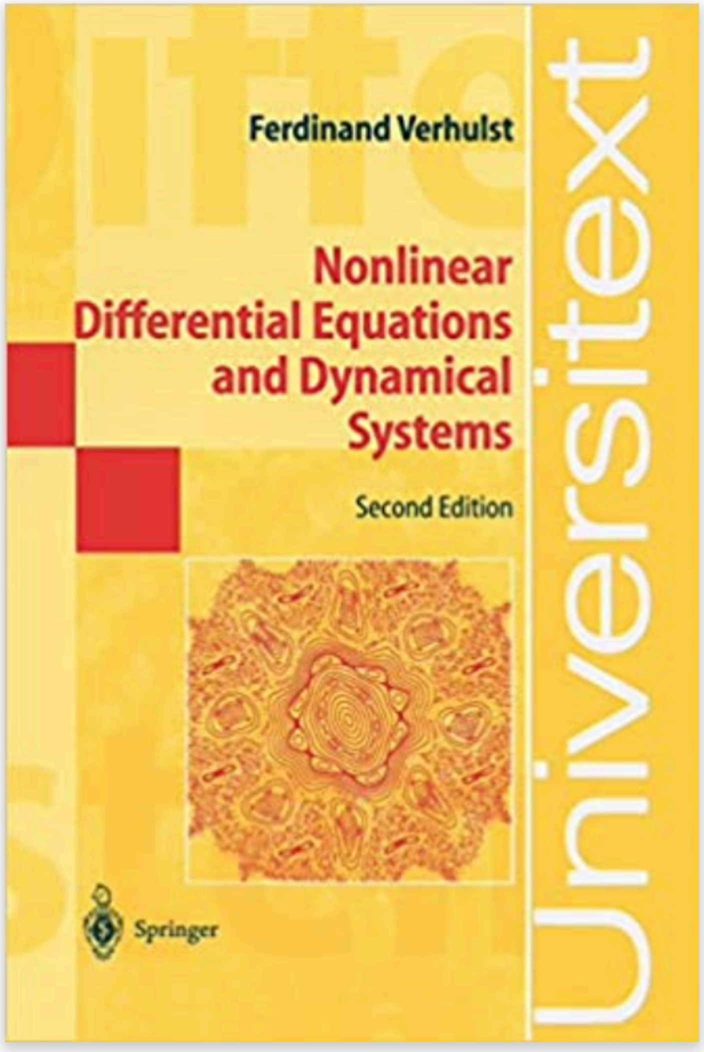
OXFORD

Nonlinear Ordinary Differential Equations

AN INTRODUCTION FOR SCIENTISTS AND ENGINEERS

FOURTH EDITION

D. W. Jordan
P. Smith



Ferdinand Verhulst

Nonlinear Differential Equations and Dynamical Systems

Second Edition

Springer

Universitext

# Dynamical Systems

Continuous time $t \in \mathbb{R}$

$\vec{x} = (x_1, \ldots, x_n)^\top \in \mathcal{A} \subset \mathbb{R}^n$ phase space (state space)

$\vec{f} : \mathcal{A} \to \mathbb{R}^n$

Time evolution $\dot{\vec{x}} := \frac{d\vec{x}}{dt} = \vec{f}(\vec{x})$

$$\left.\begin{array}{l} \begin{cases} \dot{x}_1 = \frac{dx_1}{dt} = f_1(x_1, \ldots, x_n) \\ \vdots \\ \dot{x}_n = \frac{dx_n}{dt} = f_n(x_1, \ldots, x_n) \end{cases} \\ \vec{x}(t_0) = \vec{x}_0 \quad \text{initial condition} \end{array}\right\} \quad \text{Cauchy problem}$$

# Dynamical Systems

Continuous time $t \in \mathbb{R}$

$\vec{f} : \mathcal{A} \to \mathbb{R}^n$ Sufficiently regular to ensure existence and uniqueness of the solution $\vec{\varphi}(t; \vec{x}_0)$ of the Cauchy problem

f Lipschitz is enough [Picard theorem]

# Dynamical Systems

Continuous time $t \in \mathbb{R}$

$\vec{f} : \mathcal{A} \to \mathbb{R}^n$ Sufficiently regular to ensure existence and uniqueness of the solution $\vec{\varphi}(t;\vec{x}_0)$ of the Cauchy problem

Linear case if "only monomials in $x_i$ are present"

$$\vec{f}(\vec{x}) = \mathbf{A}\vec{x} \qquad \mathbf{A} \in \mathbb{R}^{n\times n} \qquad \vec{\varphi}(t;\vec{x}_0) = e^{\mathbf{A}(t-t_0)}\vec{x}_0$$

$$e^{\mathbf{A}t} := \sum_{n\geq 0} \frac{t^n}{n!}\mathbf{A}^n$$

Non-Linear case, all the remaining ones

# Dynamical Systems

Continuous time $t \in \mathbb{R}$

$\vec{f} : \mathcal{A} \to \mathbb{R}^n$ Sufficiently regular to ensure existence and uniqueness of the solution $\vec{\varphi}(t;\vec{x}_0)$ of the Cauchy problem

Linear case if "only monomials in $x_i$ are present"

$$\vec{f}(\vec{x}) = \mathbf{A}\vec{x} \qquad \mathbf{A} \in \mathbb{R}^{n\times n} \qquad \vec{\varphi}(t;\vec{x}_0) = e^{\mathbf{A}(t-t_0)}\vec{x}_0$$

Non-Linear case, all the remaining ones

Autonomous vs non-autonomous

$$\vec{f}(\vec{x}) \qquad \vec{f}(\vec{x}, \boxed{t})$$

explicit dependence on time

# Dynamical Systems

Discrete time $n \in \mathbb{N}$

$\vec{x} = (x_1, \dots, x_n)^\top \in \mathcal{A} \subset \mathbb{R}^n$ phase space (state space)

$\vec{f} : \mathcal{A} \to \mathbb{R}^n$

Time evolution

$$\vec{x}_{n+1} = \vec{f}(\vec{x}_n)$$

$\vec{x}_0$ initial condition

# Dynamical Systems

Discrete time $n \in \mathbb{N}$

$\vec{x} = (x_1, \dots, x_n)^\top \in \mathcal{A} \subset \mathbb{R}^n$ phase space (state space)

$\vec{f} : \mathcal{A} \to \mathbb{R}^n$

Time evolution

$$\vec{x}_{n+1} = \vec{f}(\vec{x}_n)$$

$\vec{x}_0$ initial condition

IT CAN BE STUDIED
BUT WE WILL NOT DO IT

# Dynamical Systems

## Equilibrium point :

$$\vec{x}^* \in \mathcal{A} \ : \ \vec{f}(\vec{x}^*) = 0 \Rightarrow \vec{x}(t) = \vec{x}^* \quad \forall t$$

# Dynamical Systems

## Equilibrium point :

$$\vec{x}^* \in \mathcal{A} \ : \ \vec{f}(\vec{x}^*) = 0 \Rightarrow \vec{x}(t) = \vec{x}^* \quad \forall t$$

## Stability of the equilibrium point :

$\vec{x}^*$ is (locally) stable (or Lyapunov stable) if

$$\forall \epsilon > 0 \, \exists \delta(\epsilon) > 0 \ : \forall \vec{x}_0 : |\vec{x}_0 - \vec{x}^*| < \delta$$

$$\Rightarrow \ |\vec{\varphi}(t;\vec{x}_0) - \vec{x}^*| < \epsilon \, \forall t \geq 0$$

# Dynamical Systems

Equilibrium point :

$$\vec{x}^* \in \mathcal{A} \ : \ \vec{f}(\vec{x}^*) = 0 \Rightarrow \vec{x}(t) = \vec{x}^* \quad \forall t$$

Stability of the equilibrium point :

$\vec{x}^*$ is <u>asymptotically</u> (locally) <u>stable</u> if

$$\exists \delta(\vec{x}^*) > 0 \ : \forall \vec{x}_0 : |\vec{x}_0 - \vec{x}^*| < \delta \quad \Rightarrow \quad \vec{\varphi}(t; \vec{x}_0) \underset{t \to +\infty}{\longrightarrow} \vec{x}^*$$

## Dynamical Systems

Linear stability analysis of the equilibrium point :

$$\vec{x}^* \in \mathcal{A} \ : \ \vec{f}(\vec{x}^*) = 0 \Rightarrow \vec{x}(t) = \vec{x}^* \quad \forall t$$

Let $\mathbf{J}(\vec{x}^*) = \frac{\partial \vec{f}}{\partial \vec{x}}(\vec{x}^*)$ be the Jacobian matrix, i.e.,

$$J_{ij}(\vec{x}^*) = \frac{\partial f_i}{\partial x_j}(\vec{x}^*)$$

# Dynamical Systems

Linear stability analysis of the equilibrium point :

$$\vec{x}^* \in \mathcal{A} \ : \ \vec{f}(\vec{x}^*) = 0 \Rightarrow \vec{x}(t) = \vec{x}^* \quad \forall t$$

Let $\mathbf{J}(\vec{x}^*) = \frac{\partial \vec{f}}{\partial \vec{x}}(\vec{x}^*)$ be the Jacobian matrix, i.e.,

$$J_{ij}(\vec{x}^*) = \frac{\partial f_i}{\partial x_j}(\vec{x}^*)$$

Let $\lambda_1, \dots, \lambda_n$ be the eigenvalues of $\mathbf{J}(\vec{x}^*)$ then

if $\Re\lambda_i < 0 \ \forall i = 1, \dots, n$ then $\vec{x}^*$ is asymptotically stable

The case of multiple eigenvalues can be handled as well

# Dynamical Systems

Attention to non-normal matrices

A matrix $\mathbf{A} \in \mathbb{R}^{n\times n}$ is non-normal iff $\mathbf{A}\mathbf{A}^\top \neq \mathbf{A}^\top\mathbf{A}$

This implies that $\mathbf{A}$ cannot be diagonalised with orthogonal vectors

# Dynamical Systems

## Attention to non-normal matrices

A matrix $\mathbf{A} \in \mathbb{R}^{n\times n}$ is non-normal iff $\mathbf{A}\mathbf{A}^\top \neq \mathbf{A}^\top\mathbf{A}$

This implies that $\mathbf{A}$ cannot be diagonalised with orthogonal vectors

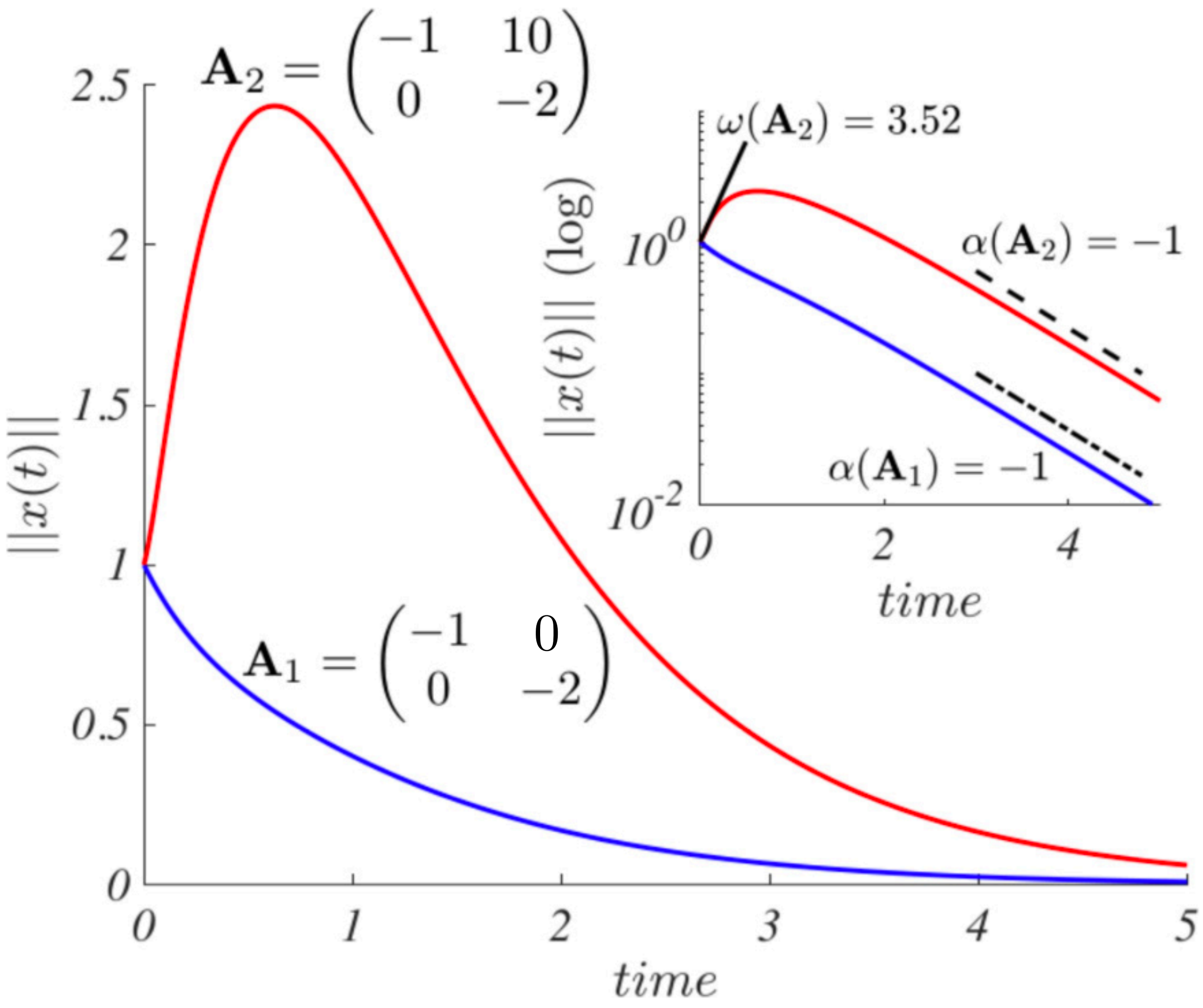
$$\frac{d\vec{x}}{dt} = \mathbf{A}\vec{x} \qquad ||\vec{x}||^2 = \vec{x}^\top \cdot \vec{x}$$

$$\alpha(\mathbf{A}) = \sup \Re\sigma(\mathbf{A})$$

spectral abscissa

$$\omega(\mathbf{A}) = \sup \sigma\left(\frac{\mathbf{A}+\mathbf{A}^\top}{2}\right)$$

numerical abscissa (reactivity)



timoteo.carletti@unamur.be

# Dynamical Systems

Solution

$$\vec{x}_0 \in \mathcal{A} \ : \ \frac{d\vec{\psi}(t;\vec{x}_0)}{dt} = \vec{f}(\vec{\psi}(t;\vec{x}_0)) \quad \forall t$$

Stability of a solution :

A solution $\vec{\psi}(t;\vec{x}_0)$ is said orbitally stable (or Poincaré stable)

$$\forall \epsilon > 0\, \exists \delta(\epsilon) > 0 \ : \forall \vec{x}_0 : |\vec{x}_0 - \vec{x}_0'| < \delta$$

$$\Rightarrow |\vec{\varphi}(t;\vec{x}_0') - \vec{\psi}(t;\vec{x}_0)| < \epsilon \, \forall t \geq 0$$

# T - Periodic solution

$$\vec{x}_0 \in \mathcal{A} \ : \ \frac{d\vec{\psi}(t;\vec{x}_0)}{dt} = \vec{f}(\vec{\psi}(t;\vec{x}_0)) \quad \forall t$$

$$\vec{\psi}(t;\vec{x}_0) = \vec{\psi}(t+T;\vec{x}_0)\,\forall t$$

timoteo.carletti@unamur.be

# Dynamical Systems

## T - Periodic solution

$$\vec{x}_0 \in \mathcal{A} \ : \ \frac{d\vec{\psi}(t;\vec{x}_0)}{dt} = \vec{f}(\vec{\psi}(t;\vec{x}_0)) \quad \forall t$$

$$\vec{\psi}(t;\vec{x}_0) = \vec{\psi}(t+T;\vec{x}_0) \, \forall t$$

## Poincaré map

Let $V$ be a (n-1)-dimensional manifold transverse to the flow of the ODE

st $\vec{x}_0 \in V$, then $\vec{\psi}(T;\vec{x}_0) = \vec{x}_0 \in V$. Let $\vec{x}' \in V$ close (enough) to $\vec{x}_0$

then let $t' > 0$ the smallest time st $P(\vec{x}') = \vec{x}'' = \vec{\varphi}(t';\vec{x}') \in V$

## Stability of T - Periodic solution

$$\forall \epsilon > 0 \, \exists \delta(\epsilon) > 0 \ : \forall \vec{x}' \ : |\vec{x}_0 - \vec{x}'| < \delta$$

$$\Rightarrow \ |P^{\circ n}(\vec{x}') - \vec{x}_0| < \epsilon \, \forall n \geq 0$$

Note : $P(\vec{x}_0) = \vec{x}_0$

## Asymptotically stability

$$|P^{\circ n}(\vec{x}') - \vec{x}_0| \underset{n \to +\infty}{\longrightarrow} 0$$

## Stability of T - Periodic solution with the Floquet theory

$$\frac{d\vec{x}}{dt} = \mathbf{A}(t)\vec{x} \qquad \mathbf{A}(t) = \mathbf{A}(t+T) \quad \forall t$$

Then there exist $\mathbf{B} \in \mathbb{R}^{n\times n}$ and $\mathbf{S}(t) \in \mathbb{R}^{n\times n}$ T-periodic and invertible

such that $\vec{x}(t) = \mathbf{S}(t)e^{\mathbf{B}t}\vec{x}_0$

Moreover $\frac{d\mathbf{S}}{dt} = \mathbf{AS} - \mathbf{SB}$

# Dynamical Systems

Let $\mathbf{C} = e^{\mathbf{B}T}$ be the monodromy matrix. The eigenvalues $\rho_j$ of $\mathbf{C}$ are called characteristic multipliers and are related to those of $\mathbf{B}$, $\mu_j$

characteristic exponents $\rho_j = e^{\mu_j T}$

[Andronov-Witte Theorem] If $\Re\mu_j < 0 \quad \forall j = 2, \dots, n$ then the periodic orbit is stable.

timoteo.carletti@unamur.be

# Dynamical Systems

Let $\mathbf{C} = e^{\mathbf{B}T}$ be the monodromy matrix. The eigenvalues $\rho_j$ of $\mathbf{C}$ are called characteristic multipliers and are related to those of $\mathbf{B}$, $\mu_j$

characteristic exponents $\rho_j = e^{\mu_j T}$

[Andronov-Witte Theorem] If $\Re\mu_j < 0 \quad \forall j = 2, \dots, n$ then the periodic orbit is stable.

Note 1: $\mu_1 = 0$

Note 2: there are not explicit general ways to compute $\mathbf{S}(t)$ or $\mu_j$

Note 3: $\rho_1 \dots \rho_n = \exp\left(\int_0^T \mathrm{tr}\mathbf{A}(t)\, dt\right)$

# Dynamical Systems

Limit cycles : isolated periodic solutions

To find limit cycles is a difficult task

(Second part of) the 16th Hilbert problem :

Determine an upper bound to the number of limit cycles in a polynomial planar EDO, as function of the degree and/or the coefficients.

This number is finite (Yulii Ilyashenko and Jean Écalle, 1991-1992)

# Dynamical Systems

To find limit cycles is a difficult task

In the plane one can use the Poincaré - Bendixon theorem.

In general the Brower fixed point theorem can be used.

# Dynamical Systems

## van der Pol oscillator

$$\frac{d^2x}{dt^2} - \mu(1 - x^2)\frac{dx}{dt} + x = 0$$

State dependent non linear damping

## Small parameter case

$$|\mu| < 2$$

## van der Pol oscillator

$$\frac{d^2x}{dt^2} - \mu(1 - x^2)\frac{dx}{dt} + x = 0$$

## Liénard coordinates

$$\dot{x} = \mu\left(x - \frac{x^3}{3}\right) - y$$
$$\dot{y} = x$$

# Dynamical Systems

## van der Pol oscillator

$$\dot{x} = \mu\left(x - \frac{x^3}{3}\right) - y$$

$$\dot{y} = x$$

$(0,0)$ is the unique equilibrium

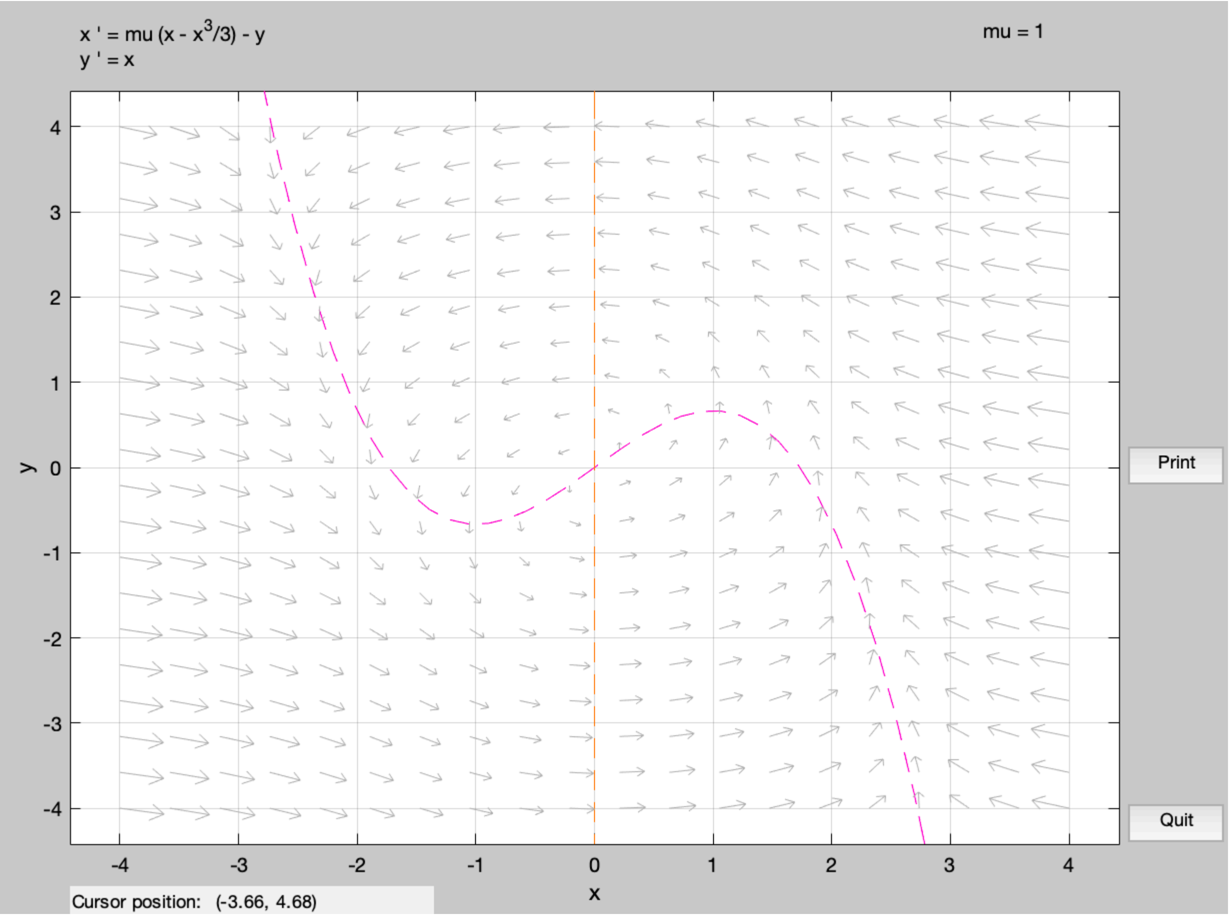
The Jacobian matrix is $\mathbf{J} = \left(\begin{smallmatrix} \mu & -1 \\ 1 & 0 \end{smallmatrix}\right)$

whose eigenvalues are

$$\lambda = \frac{\mu}{2} \pm i\sqrt{1 - \left(\frac{\mu}{2}\right)^2}$$

## 0 - isocline

$$f(x) = \mu\left(x - \frac{x^3}{3}\right)$$



x ' = mu (x - x^3/3) - y
y ' = x

mu = 1

Cursor position: (-3.66, 4.68)

# Dynamical Systems

## van der Pol oscillator

$$\dot{x} = \mu \left( x - \frac{x^3}{3} \right) - y$$

$$\dot{y} = x$$

$$\mu < 0 \Rightarrow \Re\lambda = \frac{\mu}{2} < 0$$

Stable focus

## 0 - isocline

$$f(x) = \mu \left( x - \frac{x^3}{3} \right)$$

# Dynamical Systems

van der Pol oscillator

$$\dot{x} = \mu\left(x - \frac{x^3}{3}\right) - y$$

$$\dot{y} = x$$

$$\mu > 0 \Rightarrow \Re\lambda = \frac{\mu}{2} > 0$$

Unstable focus

0 - isocline

$$f(x) = \mu\left(x - \frac{x^3}{3}\right)$$

## van der Pol oscillator

$$\dot{x} = \mu\left(x - \frac{x^3}{3}\right) - y$$

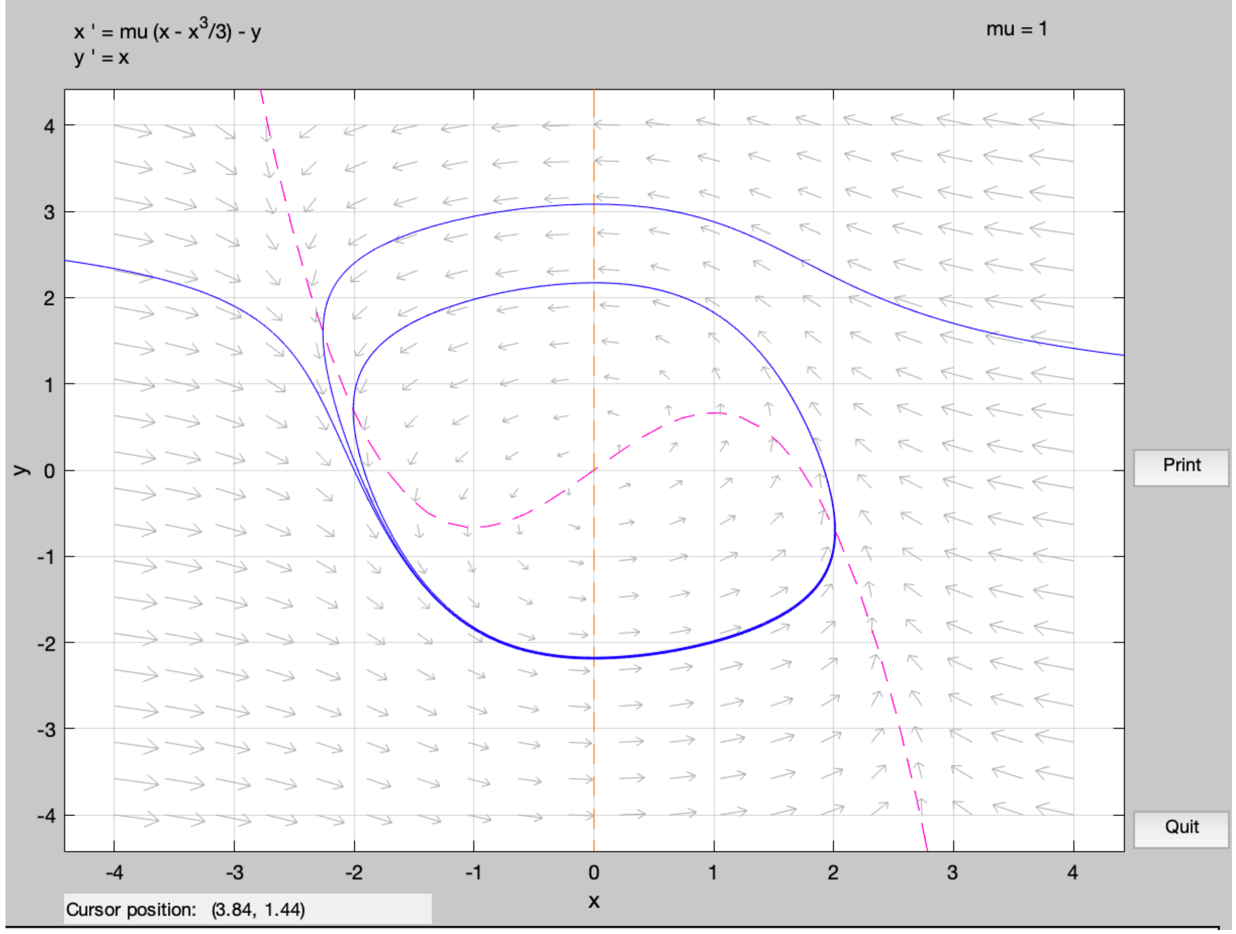
$$\dot{y} = x$$

$$\mu > 0 \Rightarrow \Re\lambda = \frac{\mu}{2} > 0$$

Unstable focus

## 0 - isocline

$$f(x) = \mu\left(x - \frac{x^3}{3}\right)$$



x ' = mu (x - x$^3$/3) - y
y ' = x

mu = 1

Cursor position: (3.84, 1.44)

# Dynamical Systems

## van der Pol oscillator

$$\dot{x} = \mu \left( x - \frac{x^3}{3} \right) - y$$

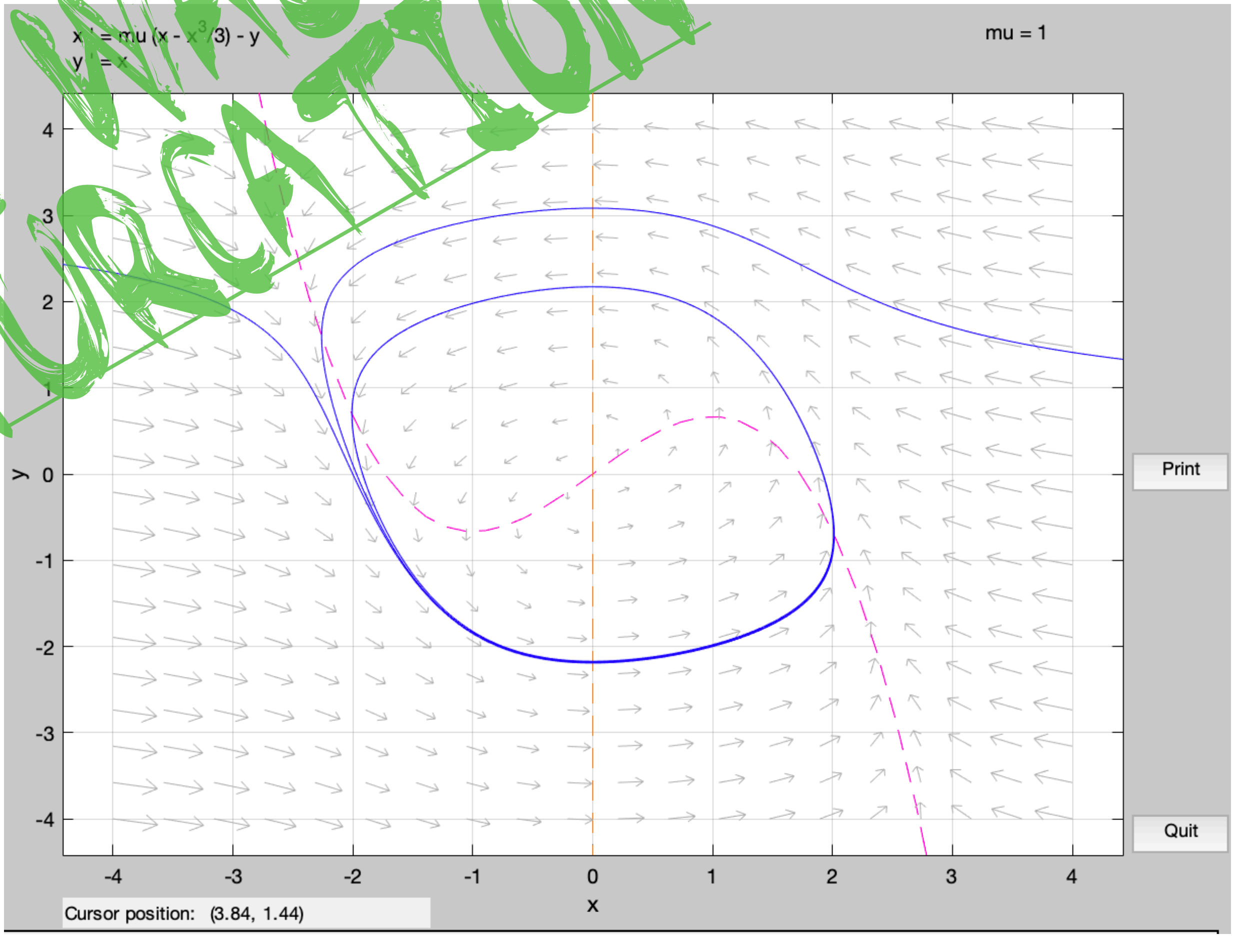
$$\dot{y} = x$$

$$\mu > 0 \Rightarrow \Re\lambda = \frac{\mu}{2} > 0$$

Unstable focus

## 0 - isocline

$$f(x) = \mu \left( x - \frac{x^3}{3} \right)$$



A LIMIT CYCLE WAS BORN
FROM A BIFURCATION

## Stuart - Landau oscillator

$$\frac{dz}{dt} = z(a + ib - |z|^2) \qquad z = x + iy \in \mathbb{C} \quad a \in \mathbb{R} \quad b \in \mathbb{R}_+$$

complex amplitude

# Dynamical Systems

## Stuart - Landau oscillator

$$\frac{dz}{dt} = z(a + ib - |z|^2) \qquad z = x + iy \in \mathbb{C} \quad a \in \mathbb{R} \quad b \in \mathbb{R}_+$$

complex amplitude

## Real variables

$$\begin{cases} \dot{x} & = ax - by - x(x^2 + y^2) \\ \dot{y} & = bx + ay - y(x^2 + y^2) \end{cases}$$

$(x, y) = (0, 0)$ Equilibrium

Stable if $a < 0$, unstable if $a > 0$

# Dynamical Systems

## Stuart - Landau oscillator

$$\frac{dz}{dt} = z(a + ib - |z|^2) \qquad z = x + iy \in \mathbb{C} \quad a \in \mathbb{R} \quad b \in \mathbb{R}_+$$

complex amplitude

Polar coordinates $z(t) = \rho(t)e^{i\theta(t)}$

$$\begin{cases} \dot{\rho} = \rho(a - \rho^2) \\ \dot{\theta} = b \end{cases}$$

A limit cycle emerges once $a$ passes from negative to positive values

# Dynamical Systems

Bifurcation.

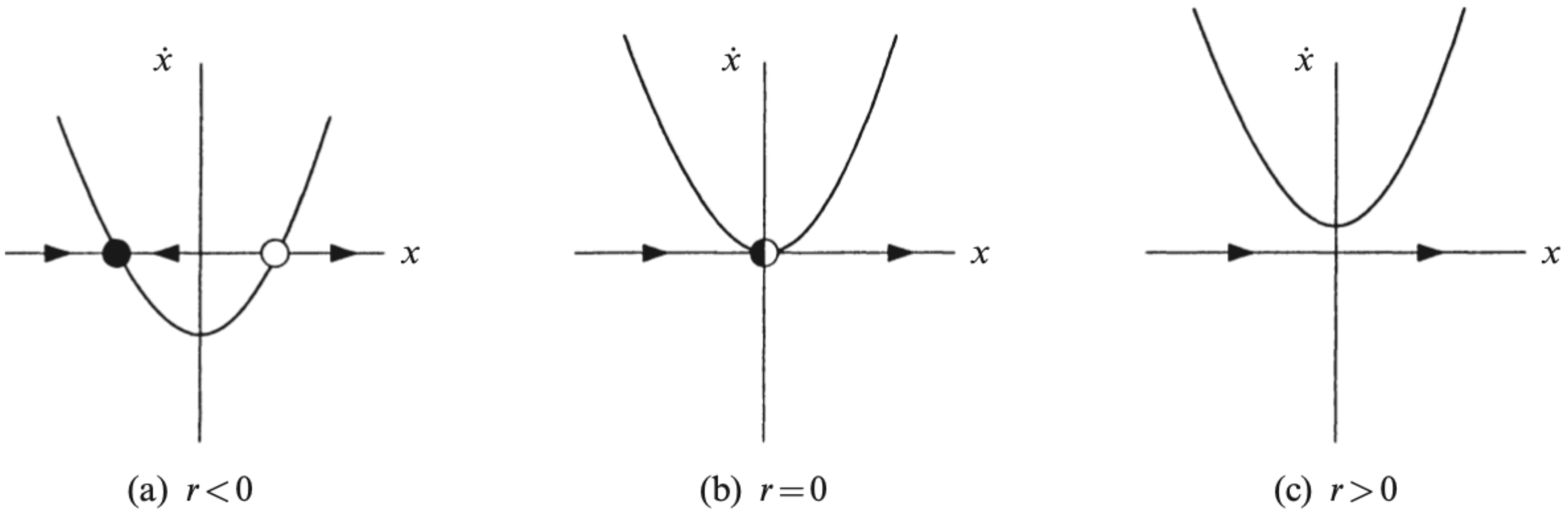
The qualitative behaviour of the system suddenly changes once a parameter reaches a critical value.

# Dynamical Systems

## Saddle - node Bifurcation.

By varying a parameter a saddle equilibrium and a stable node equilibrium merge and disappear.

$$\dot{x} = r + x^2$$

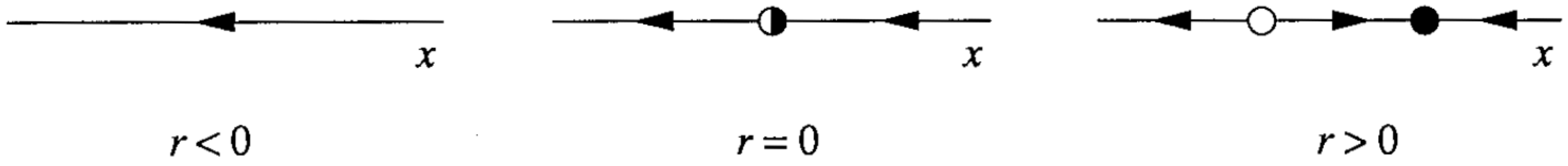


(a) $r<0$ (b) $r=0$ (c) $r>0$

# Saddle - node (blue sky) Bifurcation.

By varying a parameter a saddle equilibrium and a node equilibrium appear "out of the clear blue sky"

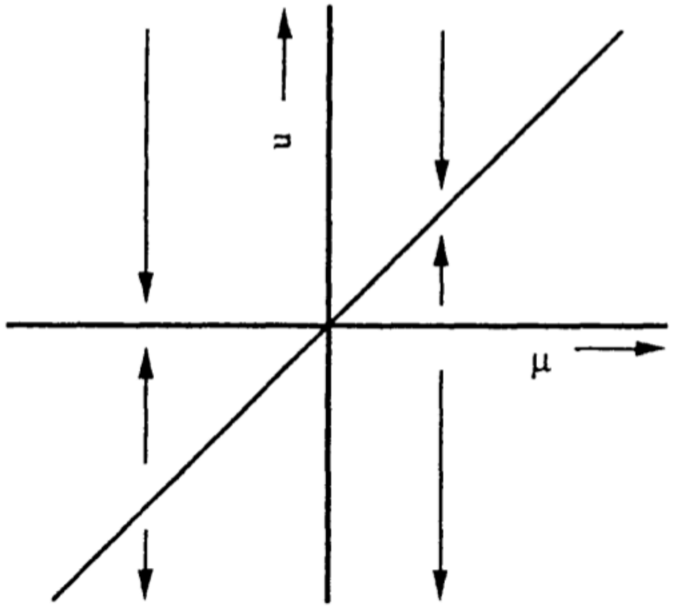
$$\dot{x} = r - x^2$$

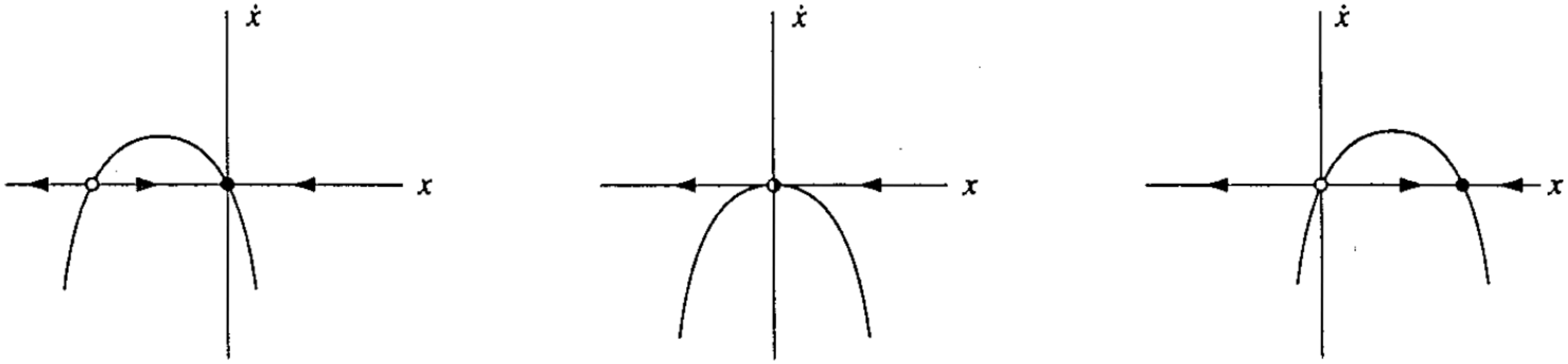


# Transcritical Bifurcation.

The equilibrium (here x=0) always exits but it changes its character by varying a parameter.

$$\dot{x} = rx - x^2$$



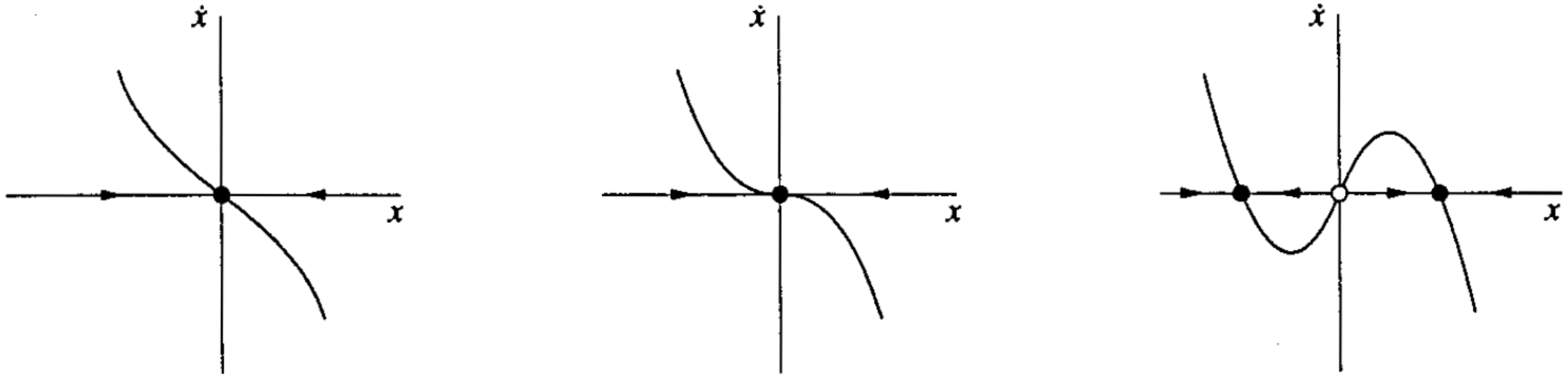


(a) $r<0$ (b) $r=0$ (c) $r>0$

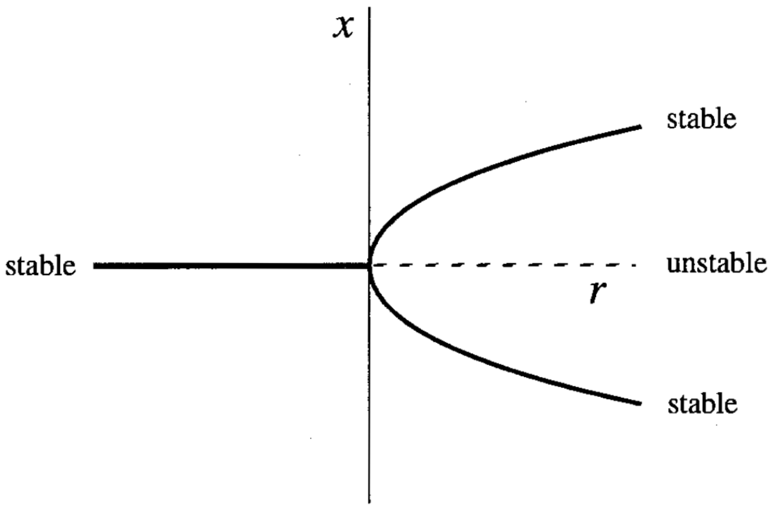
# Supercritical Pitchfork Bifurcation.

By varying a parameter a stable equilibrium (here x=0) becomes unstable and two new stable equilibria emerge.

$$\dot{x} = rx - x^3$$



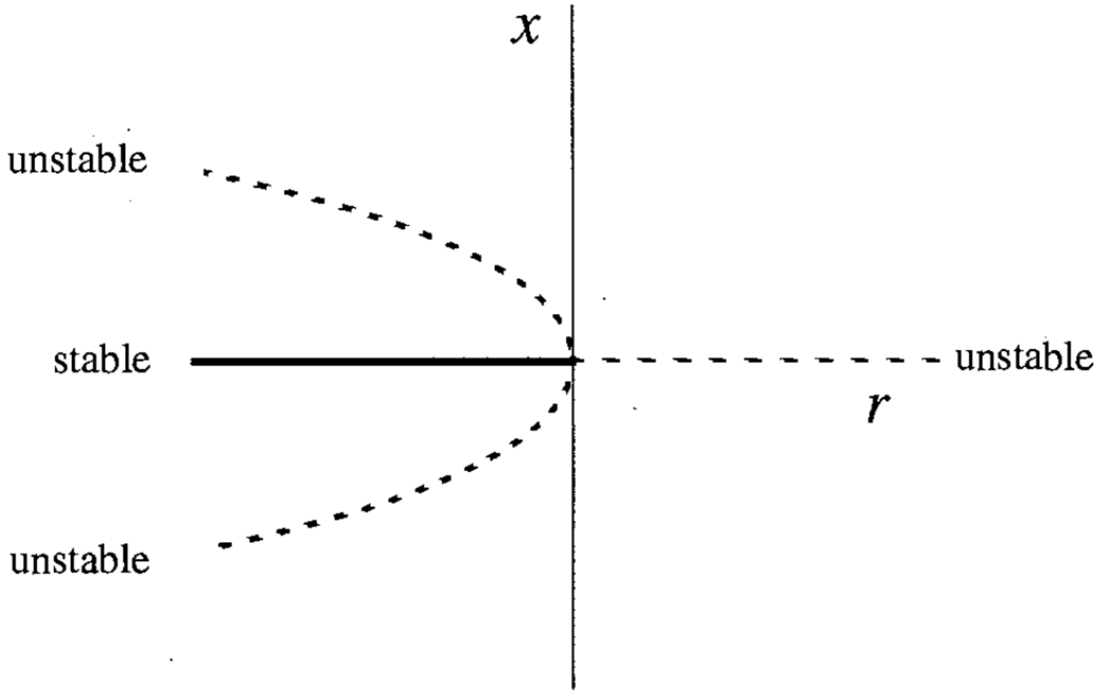
(a) $r<0$ (b) $r=0$ (c) $r>0$



# Subcritical Pitchfork Bifurcation.

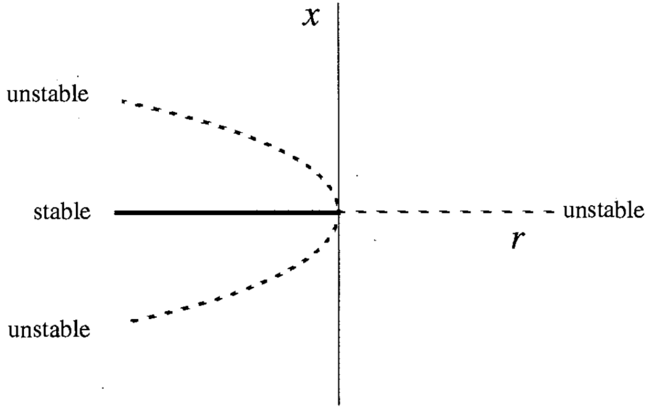
By varying a parameter a stable equilibrium (here x=0) and two unstable ones, merge and make the original equilibrium to be unstable

$$\dot{x} = rx + x^3$$



# Subcritical Pitchfork Bifurcation.

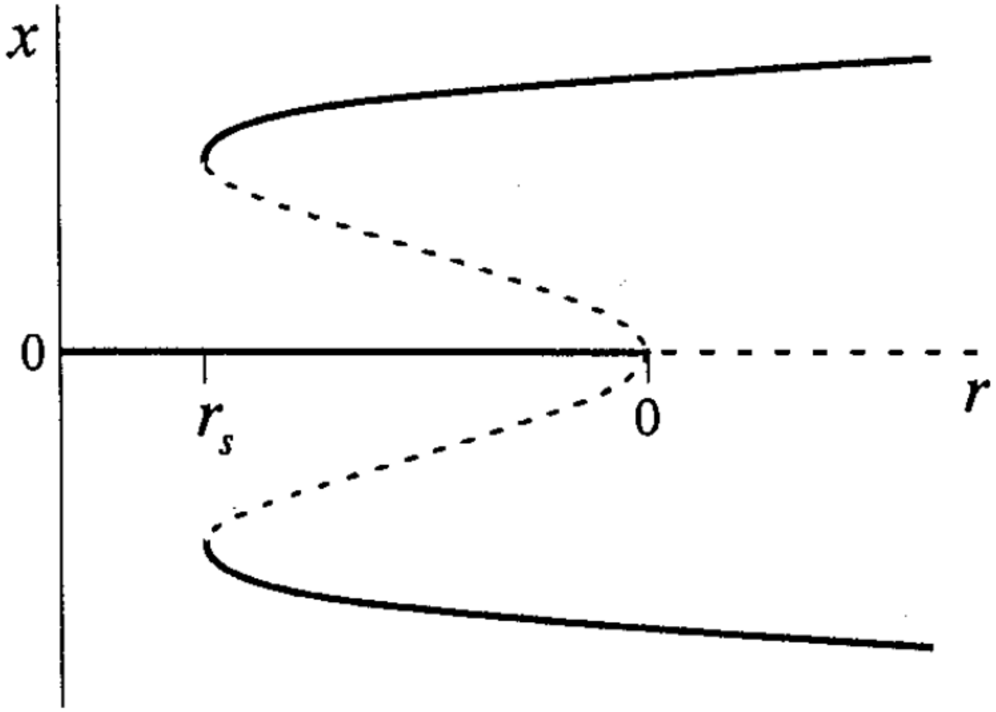
By varying a parameter a stable equilibrium (here x=0) and two unstable ones, merge and make the original equilibrium to be unstable



$$\dot{x} = rx + x^3$$

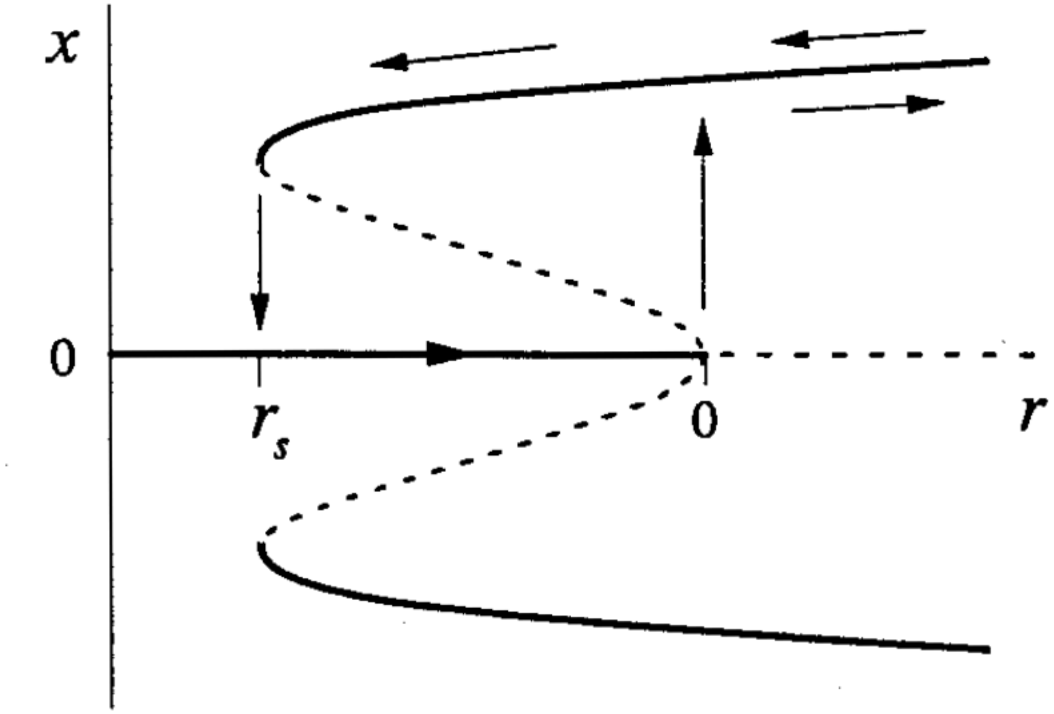
To avoid escaping orbits, one should usually introduce high order stabilising terms

$$\dot{x} = rx + x^3 - x^5$$



# Subcritical Pitchfork Bifurcation and Hysteresis

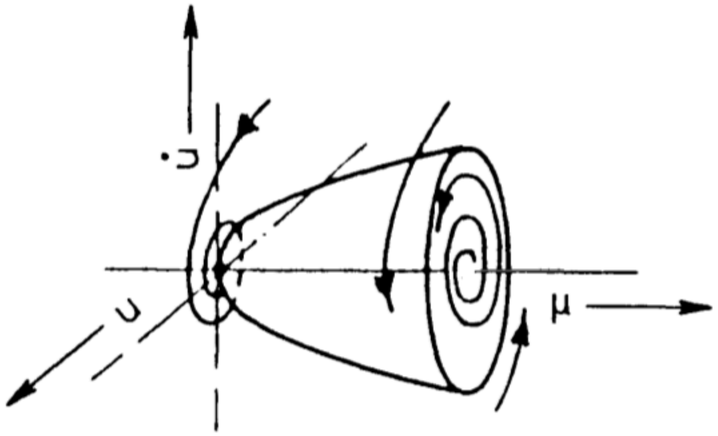
$$\dot{x} = rx + x^3 - x^5$$

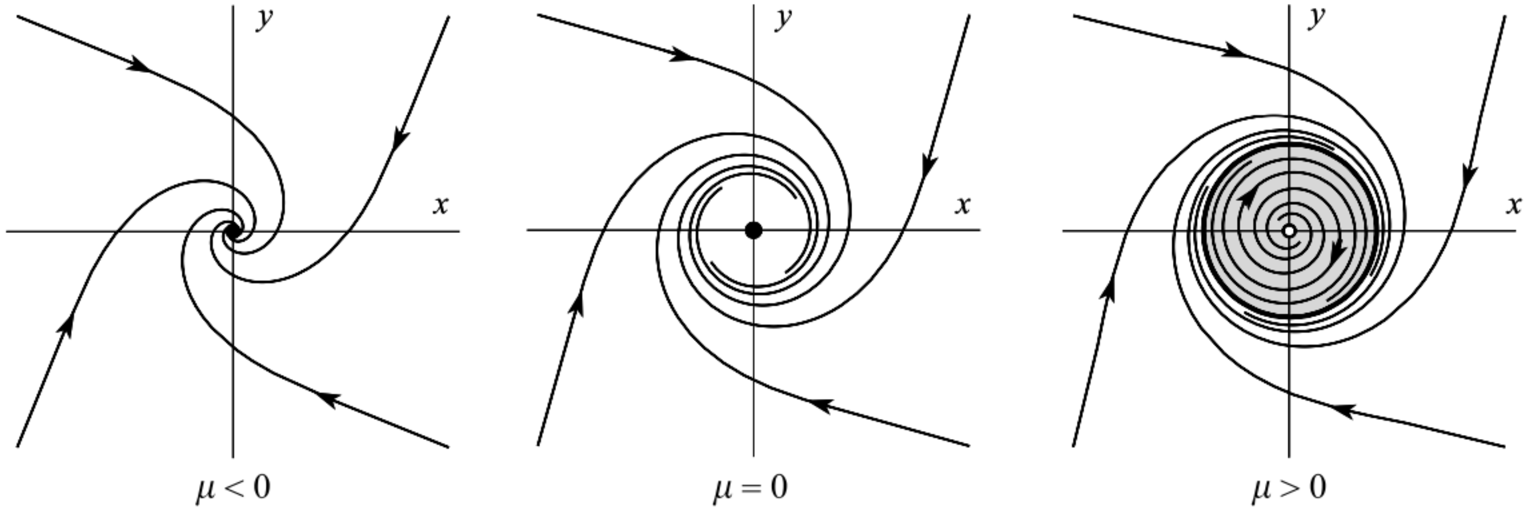


# Hopf Bifurcation.

By varying a parameter, a stable equilibrium becomes unstable and a limit cycle emerges

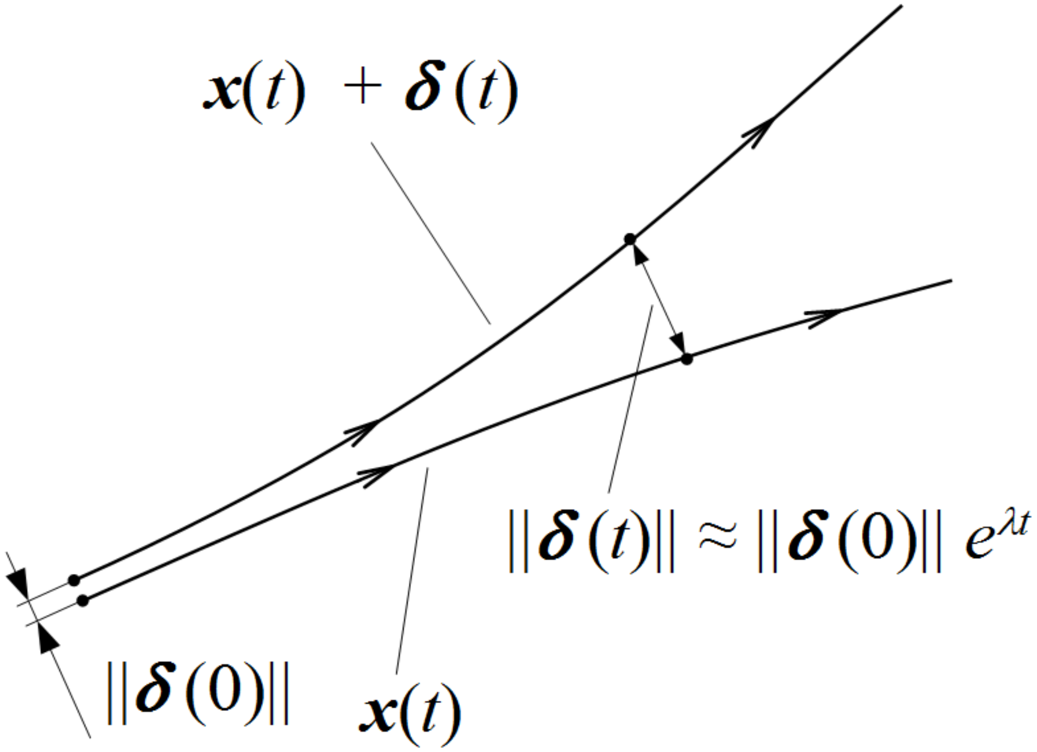
$$\begin{cases} \dot{r} &= r(\mu - r^2) \\ \dot{\theta} &= -1 \end{cases}$$





# Chaotic behaviour.

Roughly speaking, high sensitivity to initial conditions, i.e., nearby orbits diverge each others.



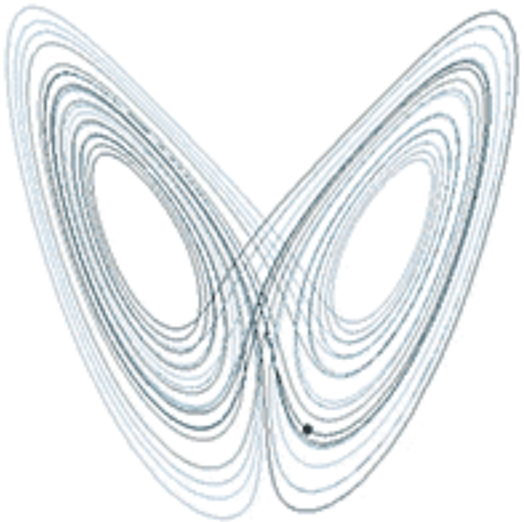
$$\lambda = \lim_{t\to\infty} \lim_{||\delta(0)||\to 0} \frac{1}{t} \log \frac{||\delta(t)||}{||\delta(0)||}$$

Maximal Lyapunov exponent

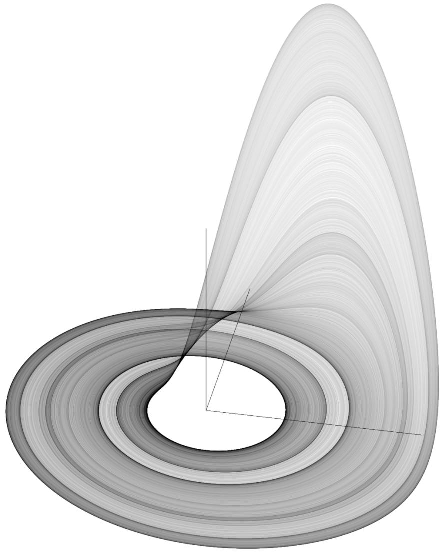
# Dynamical Systems

## Chaotic behaviour. Two “main” examples

### Lorenz system



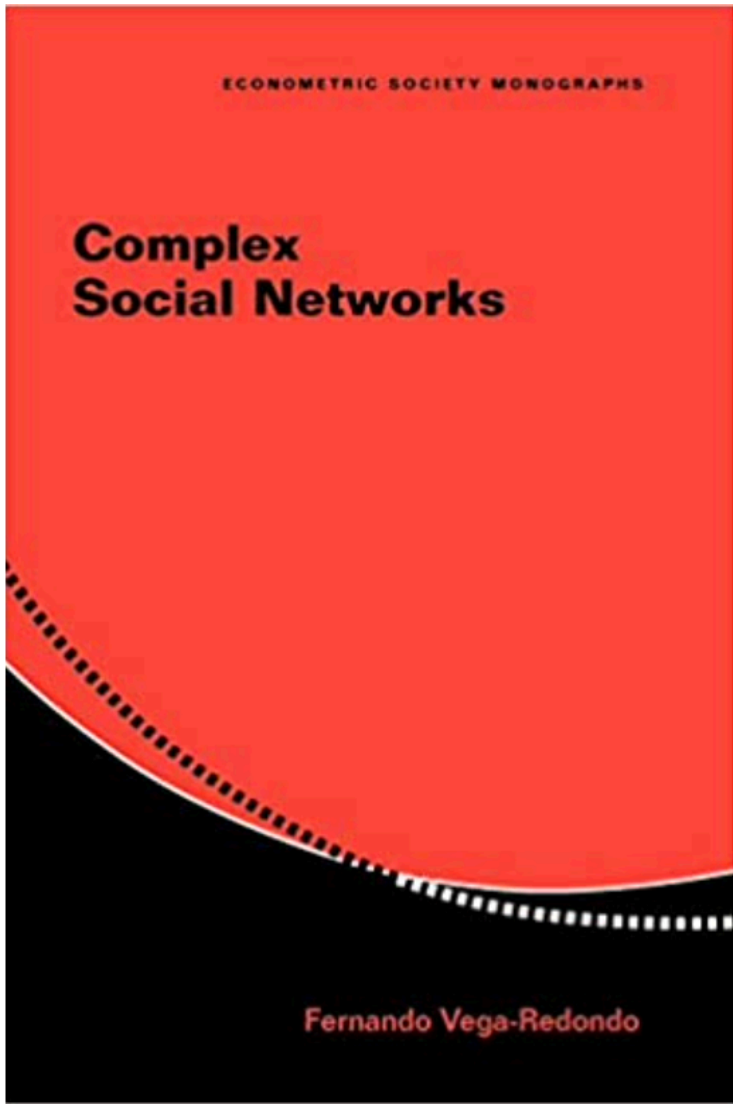
### Rössler system



$$\begin{cases} \frac{\mathrm{d}x}{\mathrm{d}t} = \sigma(y-x), \\ \frac{\mathrm{d}y}{\mathrm{d}t} = x(\rho - z) - y, \\ \frac{\mathrm{d}z}{\mathrm{d}t} = xy - \beta z. \end{cases}$$

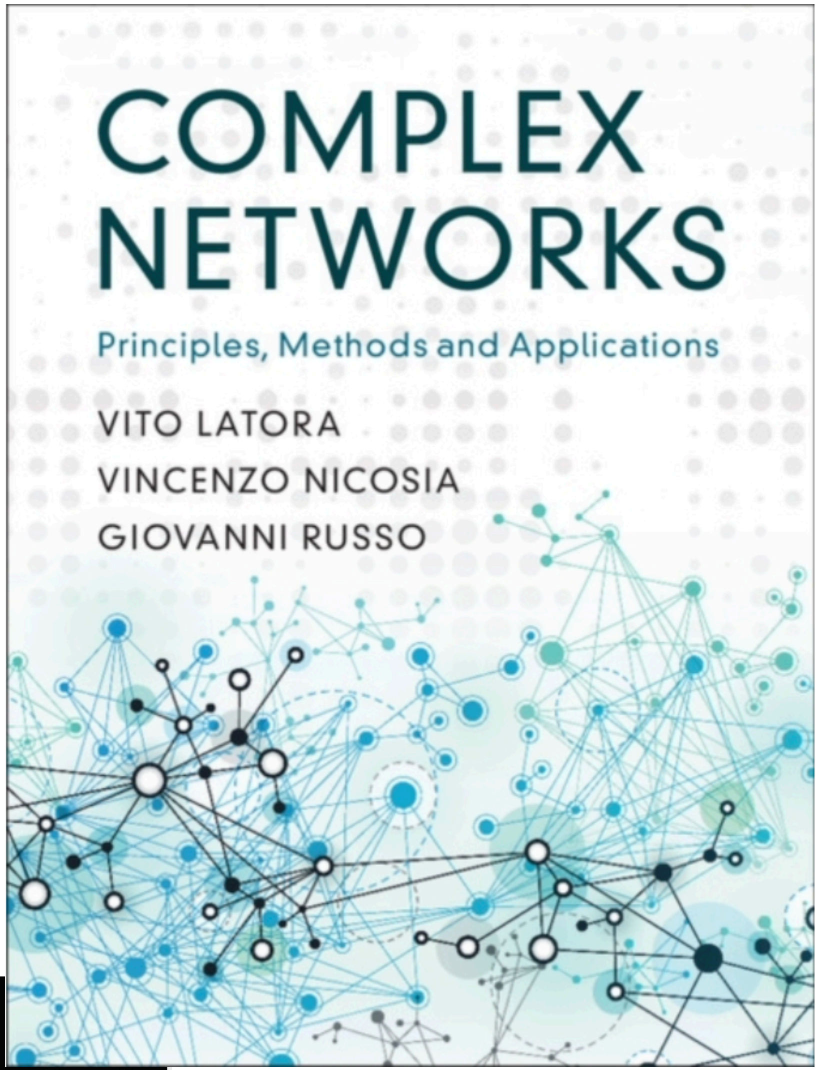
$$\begin{cases} \frac{dx}{dt} = -y - z \\ \frac{dy}{dt} = x + ay \\ \frac{dz}{dt} = b + z(x - c) \end{cases}$$

# Network theory


ECONOMETRIC SOCIETY MONOGRAPHS

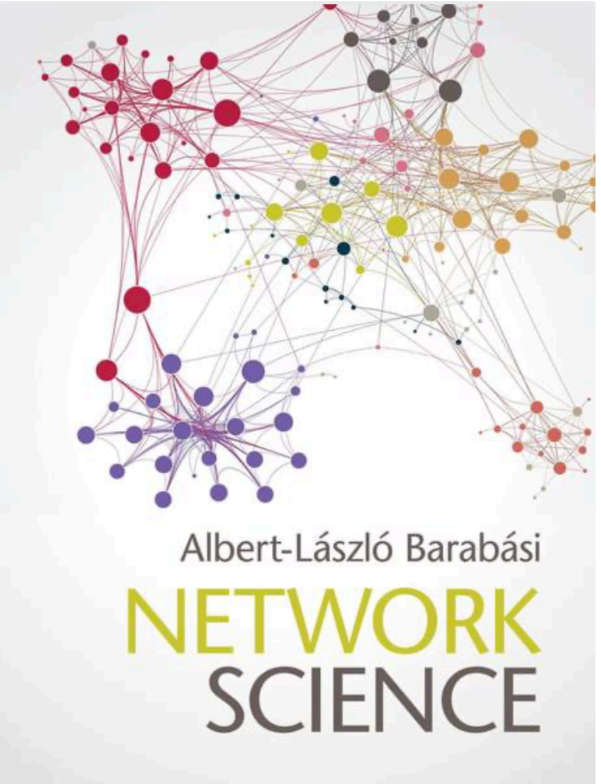
Complex
Social Networks

Fernando Vega-Redondo

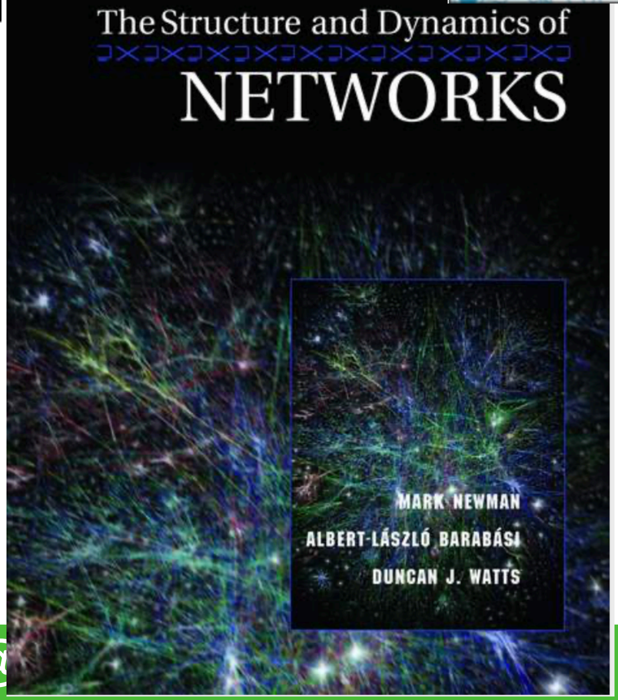

COMPLEX
NETWORKS

Principles, Methods and Applications

VITO LATORA
VINCENZO NICOSIA
GIOVANNI RUSSO


Albert-László Barabási

NETWORK
SCIENCE


The Structure and Dynamics of
NETWORKS

MARK NEWMAN
ALBERT-LÁSZLÓ BARABÁSI
DUNCAN J. WATTS

# Network theory

Non physical networks :

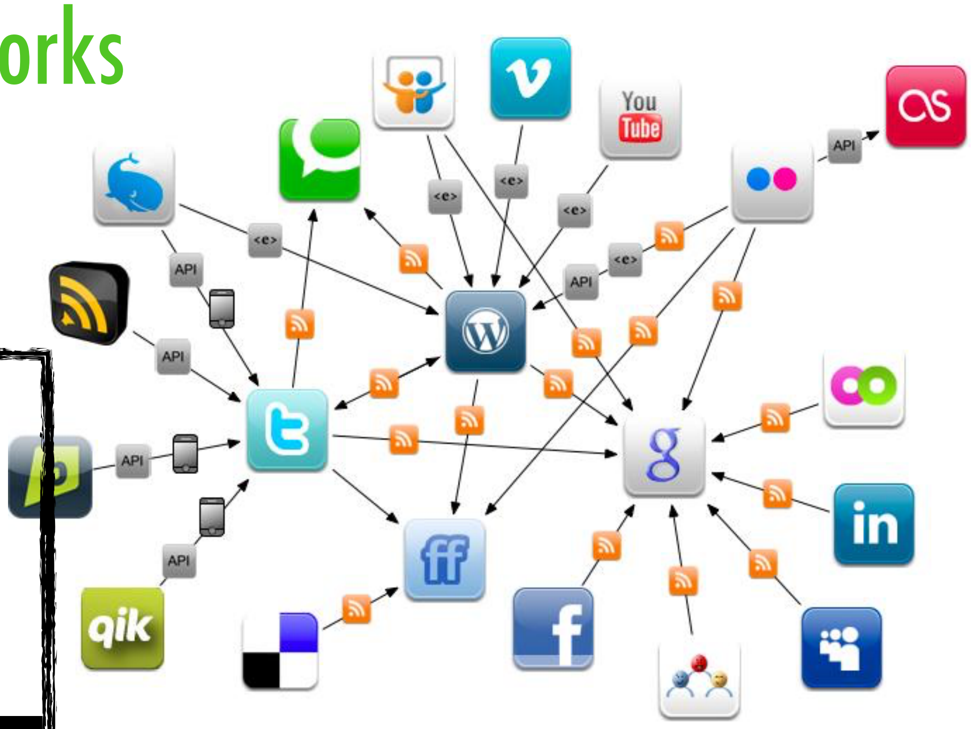
- Friendships
- www
- email / texto / call
- ...

Physical networks :

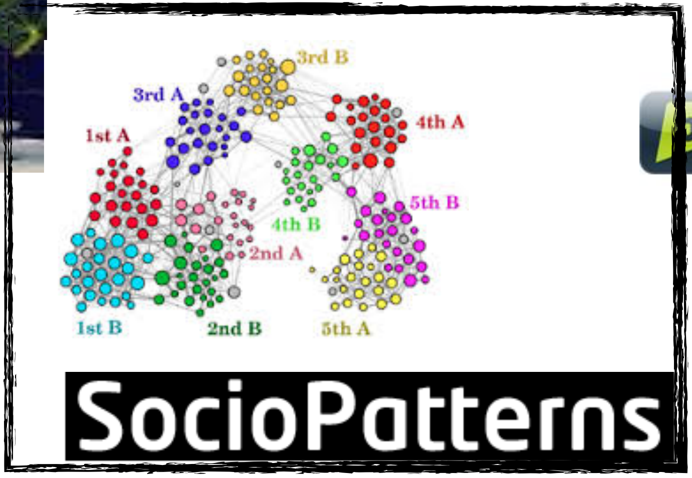
- Power plant
- internet
- road / train / flight
- ...

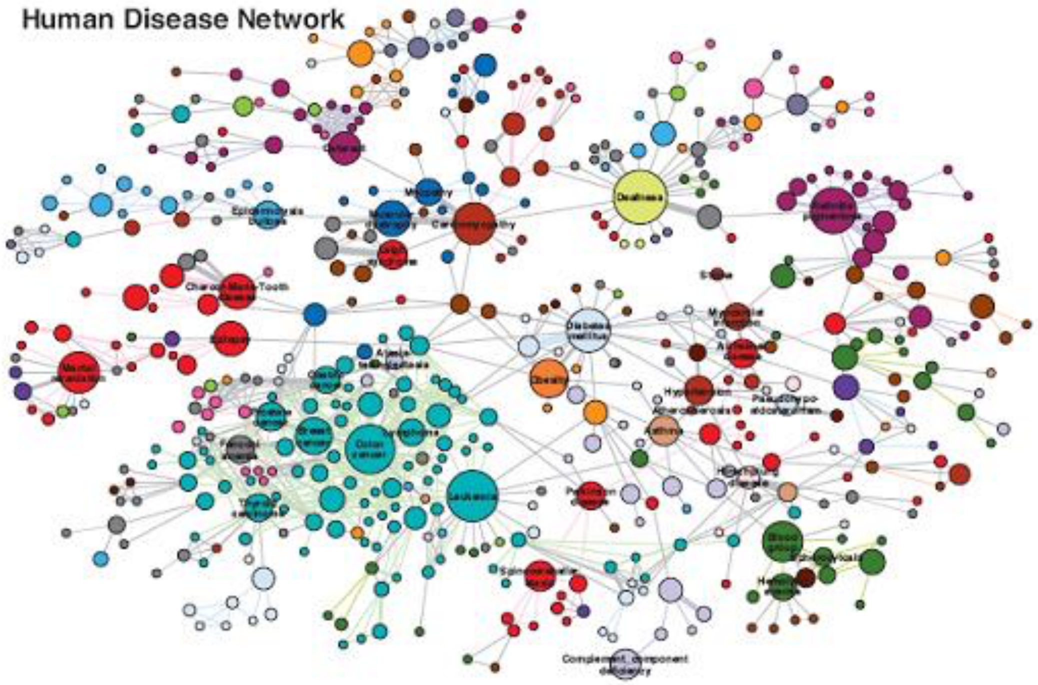
# Networks are everywhere

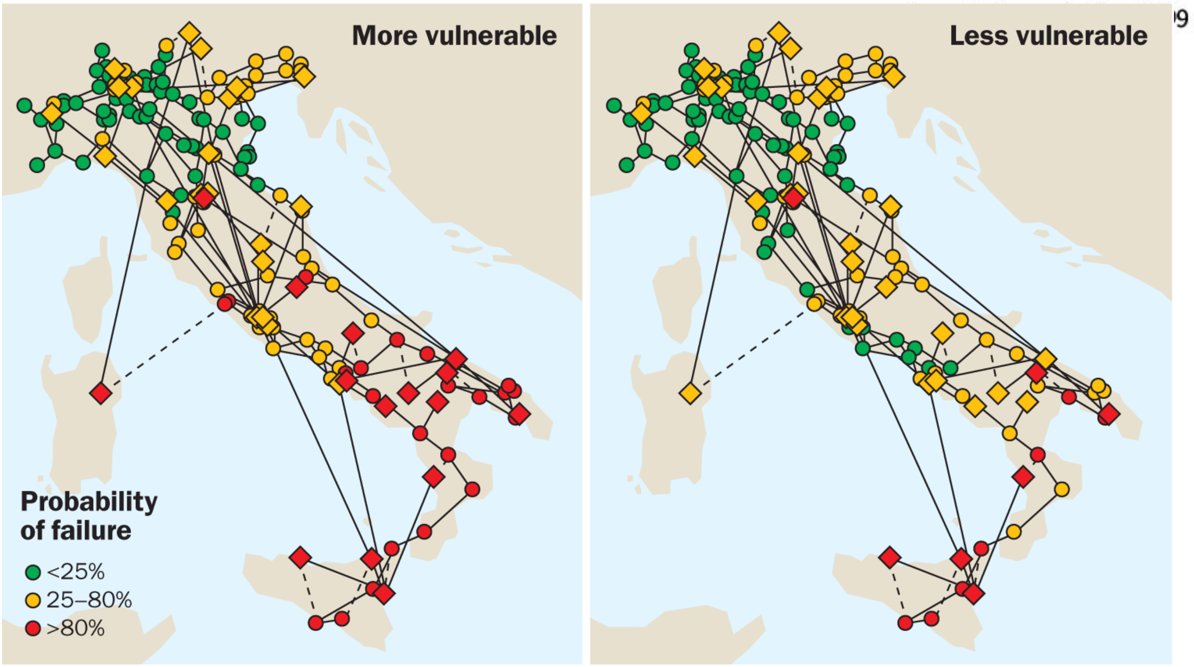
social networks



world flights map





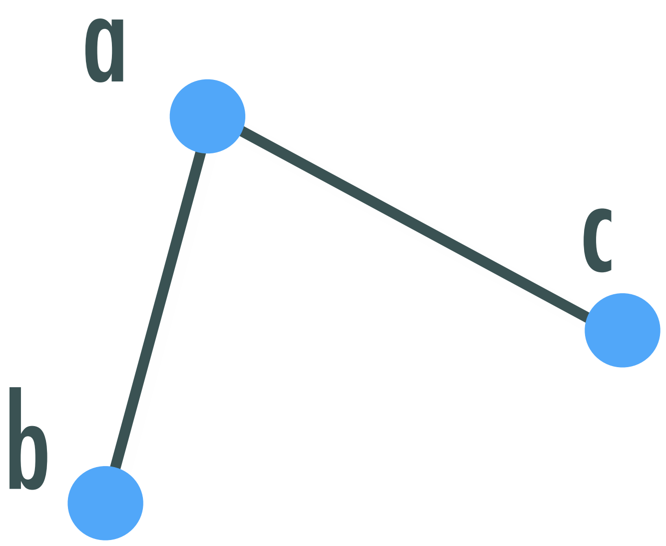


proteins networks

technological networks

# Network theory

Network = finite set of nodes pairwise connected, i.e., there is a link (edge) among the two nodes if there is some interaction among them



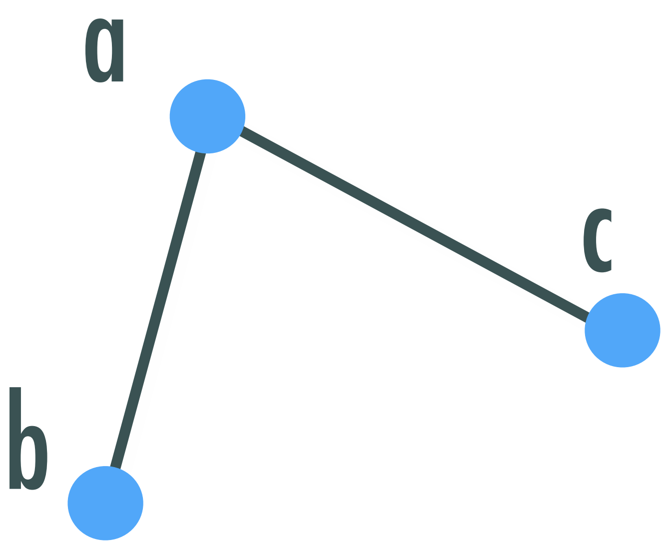
a, b and c are human beings

a and b are friends, they can exchange ideas

a and c are friends, they can exchange ideas

# Network theory

Network = finite set of nodes pairwise connected, i.e., there is a link (edge) among the two nodes if there is some interaction among them



a, b and c are <u>web pages</u>

a and b are <u>linked</u>, they share <u>hyperlinks</u>

a and c are <u>linked</u>, they share <u>hyperlinks</u>

# Network theory

A network can be encoded by the adjacency matrix $\mathbf{A} \in \{0,1\}^{n \times n}$

$$A_{ij} = 1 \quad \text{iff} \quad i \text{ and } j \text{ are linked}$$

The degree of a node is the number of its neighbours

$$k_i = \sum_{j=1}^{n} A_{ij}$$

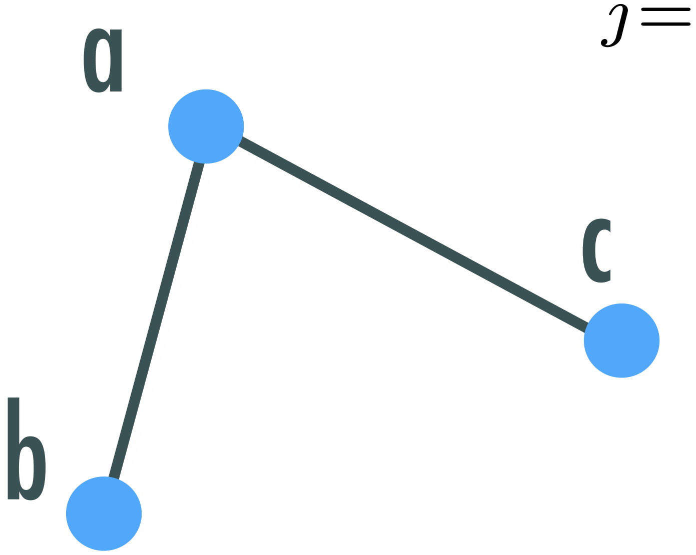
# Network theory

A network can be encoded by the adjacency matrix $\mathbf{A} \in \{0,1\}^{n\times n}$

$$A_{ij} = 1 \quad \text{iff} \quad i \text{ and } j \text{ are linked}$$

The degree of a node is the number of its neighbours

$$k_i = \sum_{j=1}^{n} A_{ij}$$



$$A_{ij} = \begin{array}{c} \begin{array}{ccc} a & b & c \end{array} \\ \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} \end{array} \qquad k_a = 2\,, k_b = 1\,, k_c = 1$$

# Network theory

If all the links are <u>reciprocal</u> ones, then we have an <u>undirected</u> network

$$A_{ij} = A_{ji} \quad \forall i, j$$

Otherwise, we have an <u>directed</u> network

$$A_{ij} = 1 \quad \text{iff} \quad j \to i$$

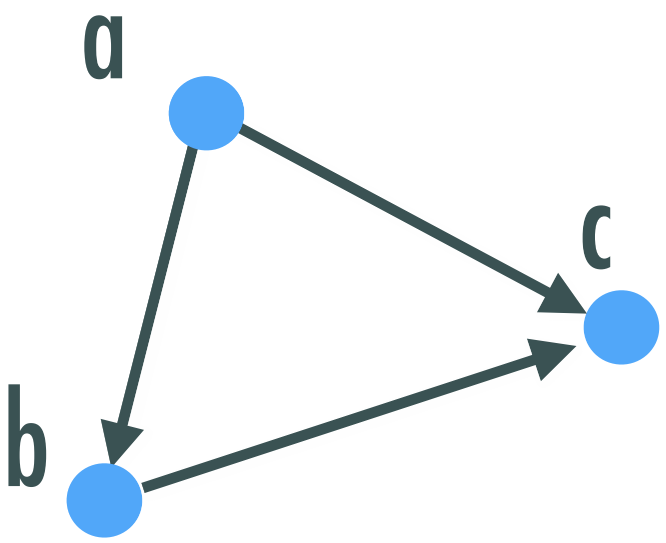
# Network theory

If all the links are reciprocal ones, then we have an undirected network

$$A_{ij} = A_{ji} \quad \forall i, j$$

Otherwise, we have an directed network

$$A_{ij} = 1 \quad \text{iff} \quad j \to i$$



$$A_{ij} = \begin{array}{c} \begin{array}{ccc} \text{a} & \text{b} & \text{c} \end{array} \\ \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 1 & 1 & 0 \end{pmatrix} \end{array}$$

# Network theory

If all the links are reciprocal ones, then we have an undirected network

$$A_{ij} = A_{ji} \quad \forall i, j$$

Otherwise, we have an directed network

$$A_{ij} = 1 \quad \text{iff} \quad j \to i$$

The out-degree of a node is the number of exiting links

$$k_i^{out} = \sum_{j=1}^{n} A_{ji}$$

The in-degree of a node is the number of entering links

$$k_i^{in} = \sum_{j=1}^{n} A_{ij}$$

# Network theory

If all the links are reciprocal ones, then we have an undirected network

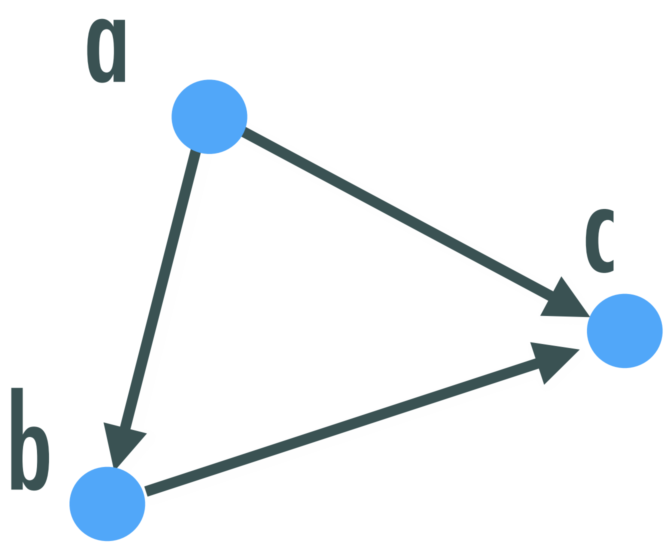
$$A_{ij} = A_{ji} \quad \forall i, j$$

Otherwise, we have an directed network

$$A_{ij} = 1 \quad \text{iff} \quad j \to i$$

The out-degree of a node is the number of exiting links

$$k_i^{out} = \sum_{j=1}^{n} A_{ji}$$



The in-degree of a node is the number of entering links

$$k_i^{in} = \sum_{j=1}^{n} A_{ij}$$

$$k_a^{out} = 2\,, k_b^{out} = 1\,, k_c^{out} = 0$$

$$k_a^{in} = 0\,, k_b^{in} = 1\,, k_c^{in} = 2$$

# Network theory

Links can be weighted $\mathbf{A} \in \mathbb{R}_+^{n\times n}$

$$A_{ij} = s \quad \text{iff} \quad i \text{ and } j \text{ have a connection whose weight is } s$$

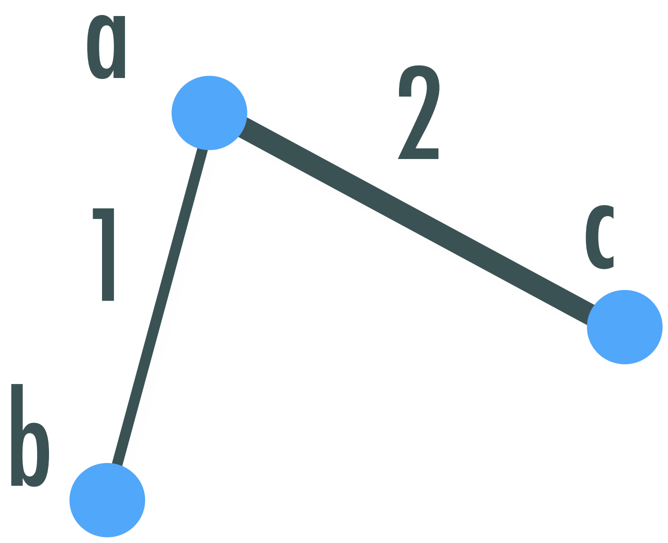
# Network theory

Links can be weighted $\mathbf{A} \in \mathbb{R}_+^{n\times n}$

$$A_{ij} = s \quad \text{iff} \quad i \text{ and } j \text{ have a connection whose weight is } s$$

The degree is replaced by the notion of strength

$$s_i = \sum_j A_{ij}$$



$$A_{ij} = \begin{pmatrix} 0 & 1 & 2 \\ 1 & 0 & 0 \\ 2 & 0 & 0 \end{pmatrix}$$

$$s_a = 3\,, s_b = 1\,, s_c = 2$$

# Network theory

A network can be also encoded by the incidence matrix $\mathbf{M} \in \{0,1\}^{m\times n}$

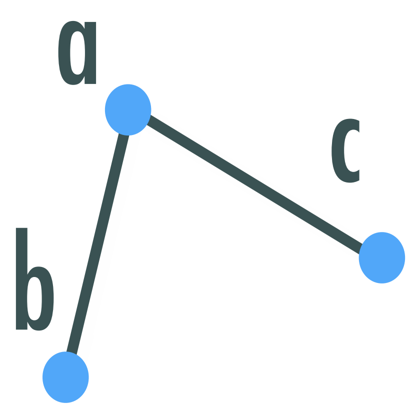
m number of links, n number of nodes

Let $e=(i,j)$ be link in the network, then

$$M_{e,i}=1\,, M_{e,j}=-1 \qquad\qquad M_{e,k}=0 \quad \forall k\neq i,j$$

# Network theory

$$e = (i,j) \qquad M_{e,i} = 1\,, M_{e,j} = -1 \qquad M_{e,k} = 0 \quad \forall k \neq i,j$$



$$A_{ij} = \begin{array}{c} \begin{array}{ccc} a & b & c \end{array} \\ \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} \end{array} \qquad M_{ij} = \begin{array}{cccc} a & b & c & \\ 1 & -1 & 0 & (a,b) \\ 1 & 0 & -1 & (a,c) \end{array}$$

$$-\mathbf{M}^\top \mathbf{M} = \begin{pmatrix} -2 & 1 & 1 \\ 1 & -1 & 0 \\ 1 & 0 & -1 \end{pmatrix}$$

$$-\mathbf{M}^\top \mathbf{M} = \mathbf{A} - \mathrm{diag}(k_1, k_2, k_3)$$

Models of networks.

## Erdős–Rényi (random network) $G(n,p)$

Given $n$ nodes, consider all pairs and with probability $p \in (0,1)$ link them.

1) The average number of links is $\binom{n}{2}p$

Erdős, P.; Rényi, A. (1959). "On Random Graphs", *Publicationes Mathematicae*, **6**, 290–297

# Network theory

Models of networks.

## Erdős–Rényi (random network) $G(n,p)$

Given $n$ nodes, consider all pairs and with probability $p \in (0,1)$ link them.

1) The average number of links is $\binom{n}{2} p$

2) The probability to have a node with degree k is

$$P(k) = \binom{n-1}{2} p^k (1-p)^{n-1-k}$$

3) if $n \to \infty$ and $np = const$ then

$$P(k) \sim \frac{(np)^k e^{-np}}{k!}$$

# Network theory

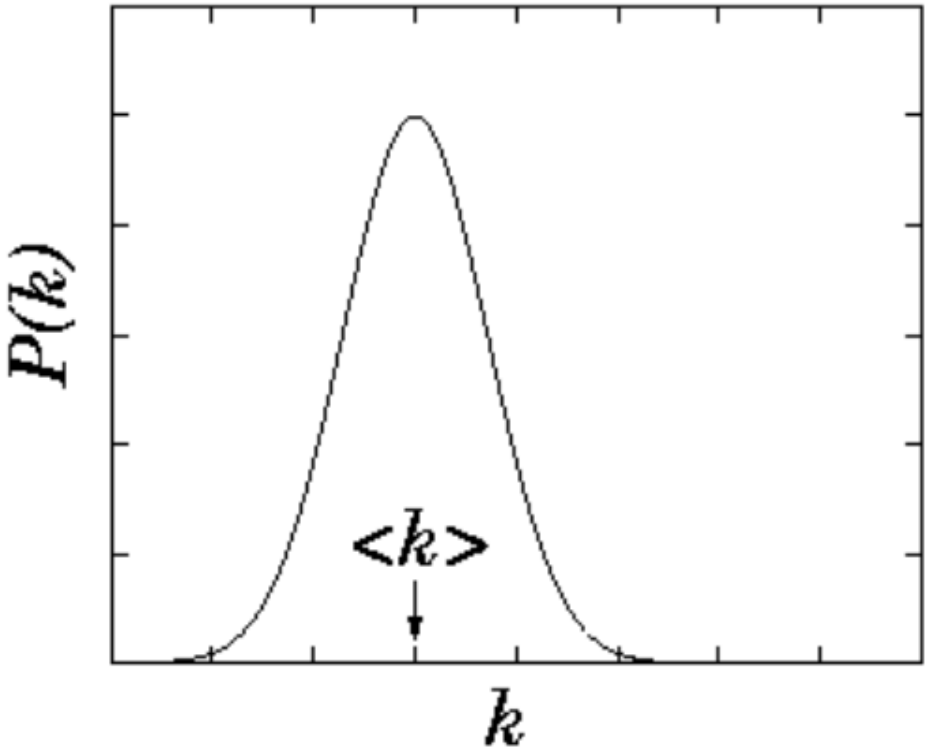
Models of networks.

## Erdős–Rényi (random network) $G(n,p)$

Given $n$ nodes, consider all pairs and with probability $p \in (0,1)$ link them.

$$n \to \infty \qquad np = const \qquad P(k) \sim \frac{(np)^k e^{-np}}{k!}$$

$$\langle k \rangle = \sum_k k P(k)$$



timoteo.carletti@unamur.be

## Models of networks.

## Watts–Strogatz (random network with small world property)

$\ell_{ij}$ Distance among two nodes = number of "hops" needed to connect them

Average shortest path : $$\ell_G = \frac{1}{n(n-1)} \sum_{i \neq j} \ell_{ij}$$

Complete network $\ell_G = 1$

d-dimensional lattice (n nodes) $\ell_G \sim n^{1/d}$

Erdős–Rényi (n,p) $$\ell_G \sim \frac{\log n}{\log(np)}$$

# Network theory

Models of networks.

## Watts-Strogatz (random network with small world property)

Let $y_i$ be the actual number of links between the neighbours of node i

the (local) clustering coefficient is

$$C_i = \frac{y_i}{k_i(k_i-1)/2}$$

# Network theory

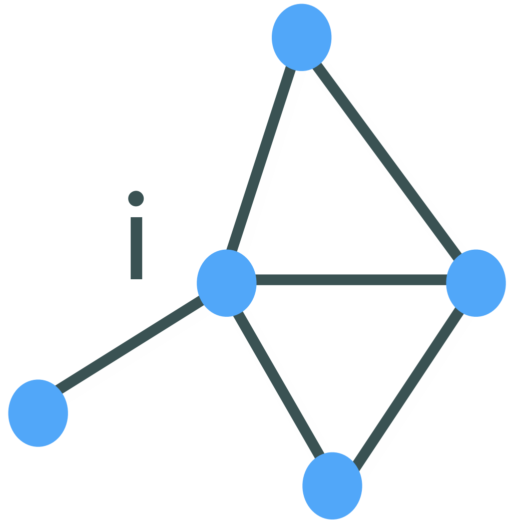
Models of networks.

## Watts–Strogatz (random network with small world property)

$$C_i = \frac{y_i}{k_i(k_i-1)/2}$$

$$k_i = 4$$

$$y_i = 2$$



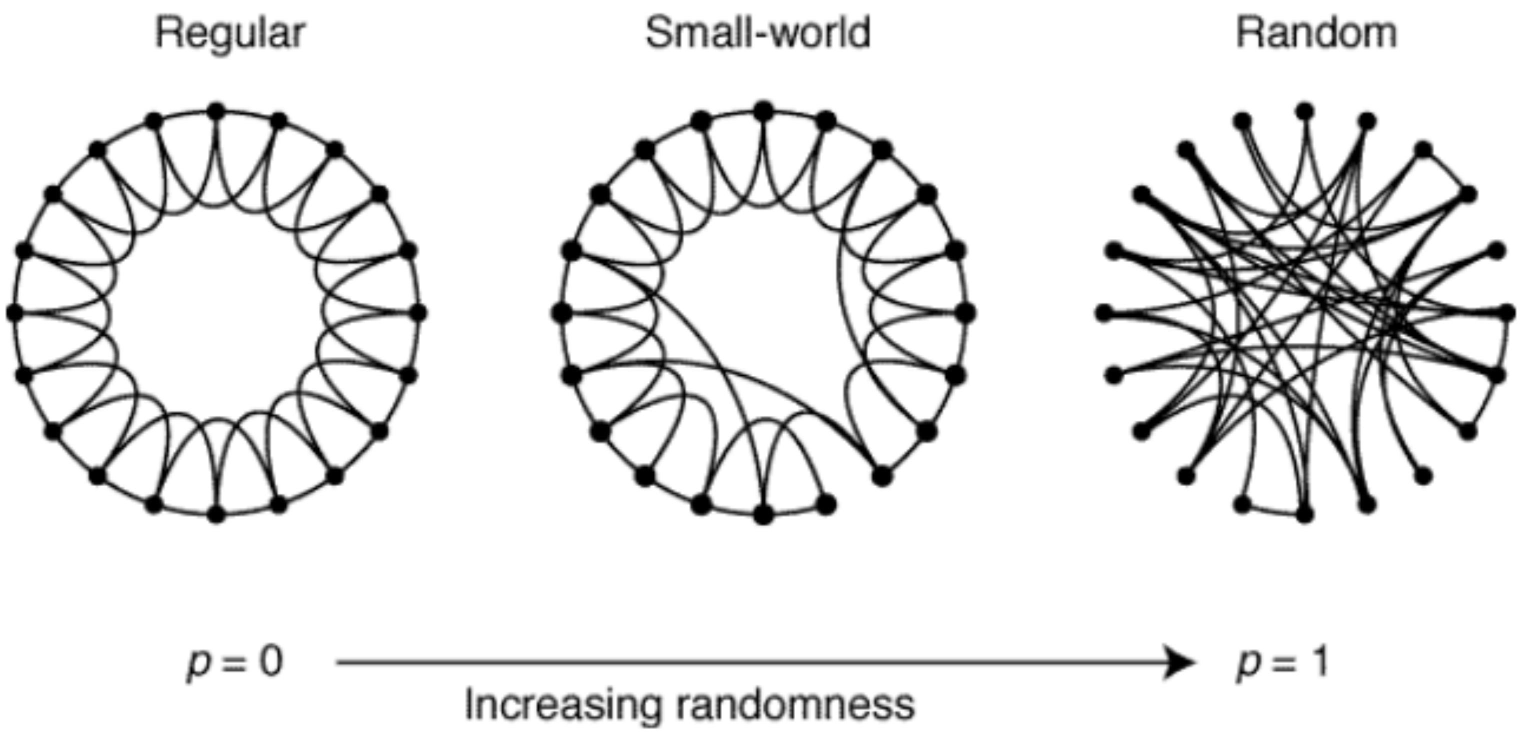
$$C_i = \frac{2}{4\times 3/2} = \frac{1}{3}$$

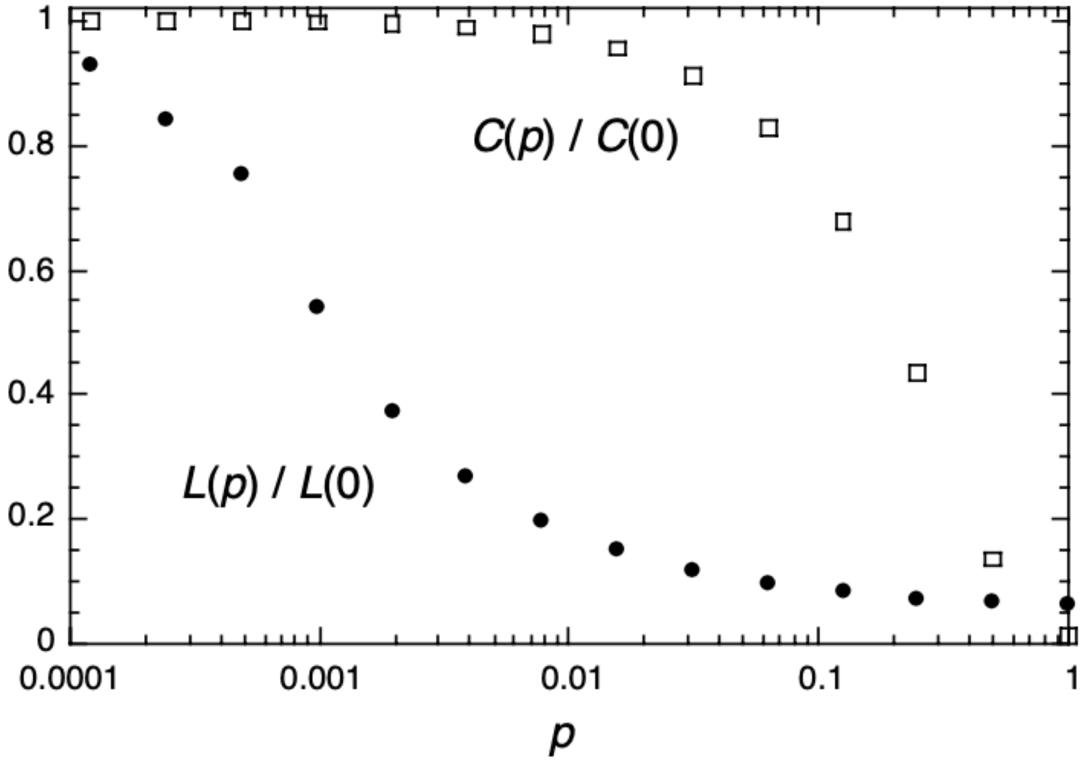
That is, the number of triangles among all the possible ones

the clustering coefficient is $C_G = \frac{1}{n}\sum_i C_i$

# Models of networks.

## Watts–Strogatz (random network with small world property)



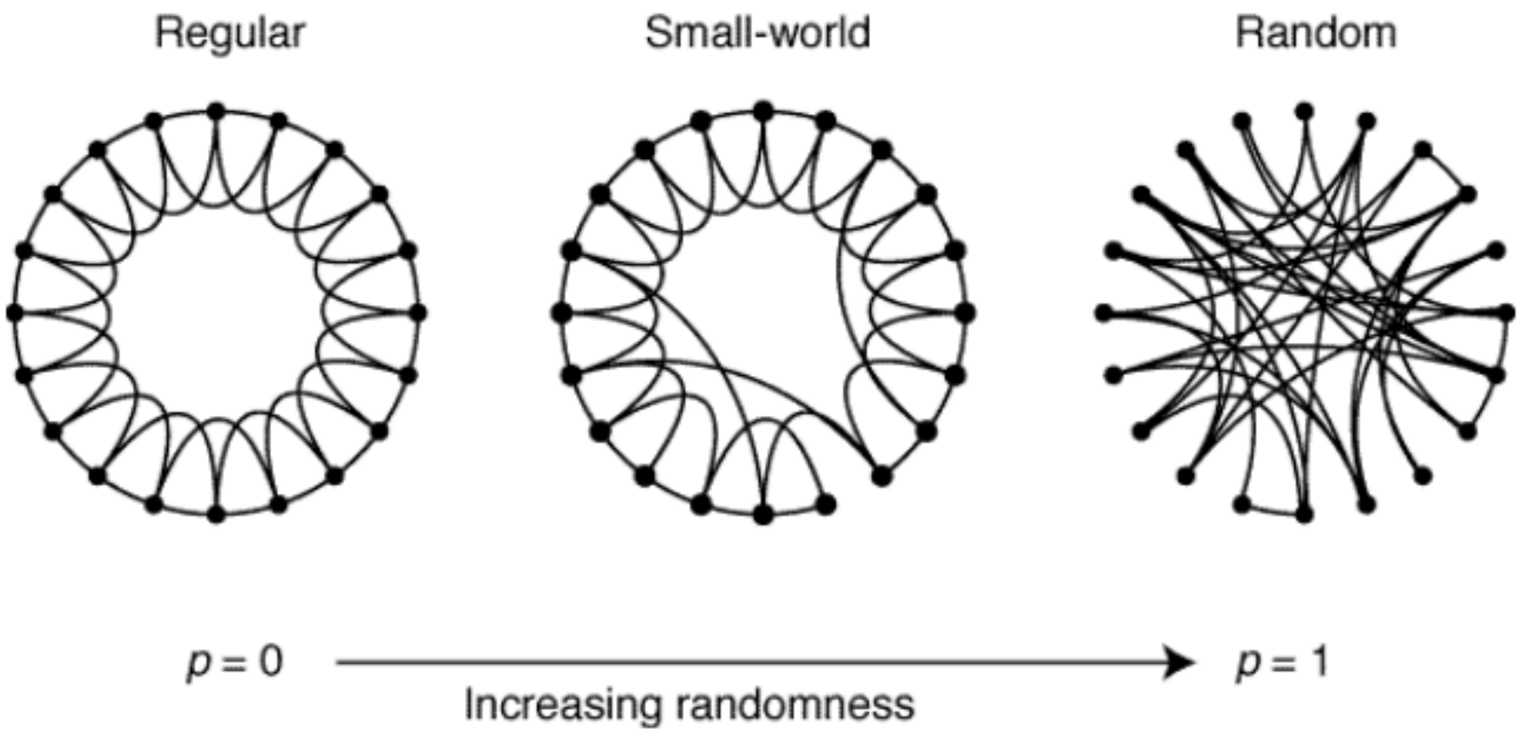


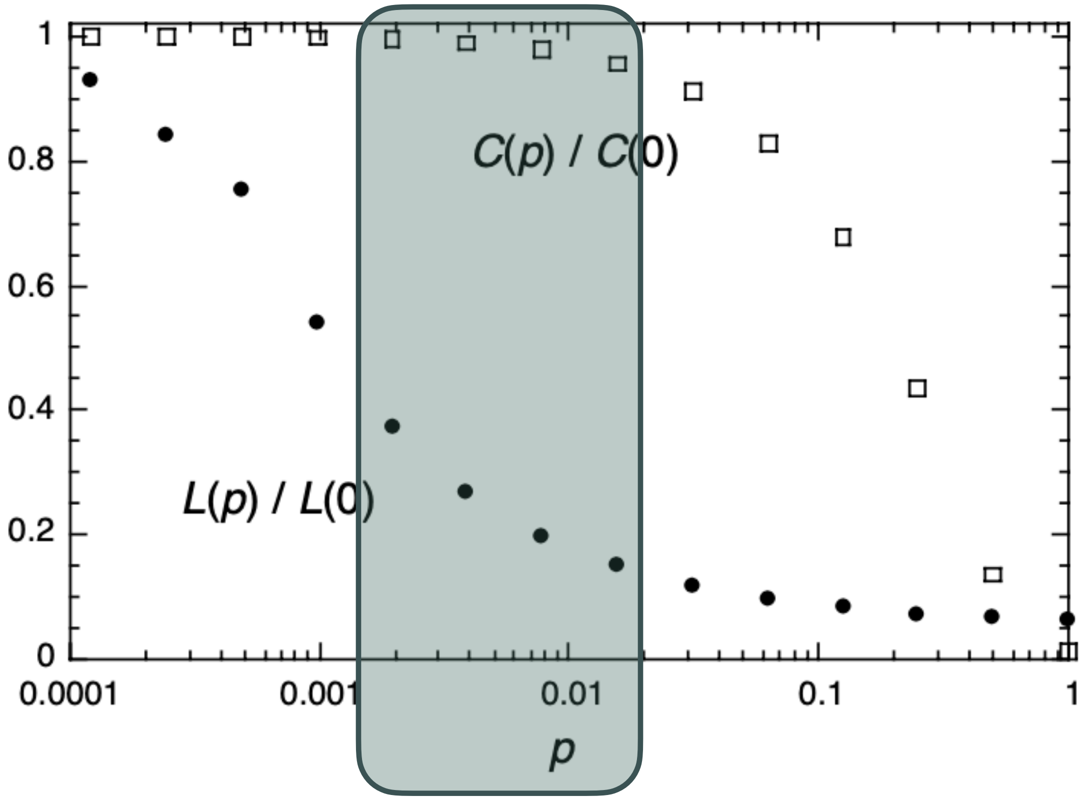
Watts, D. J.; Strogatz, S. H. (1998). “Collective dynamics of 'small-world' networks”, *Nature*, **393** (6684): 440–442.

# Network theory

## Models of networks.

## Watts–Strogatz (random network with small world property)



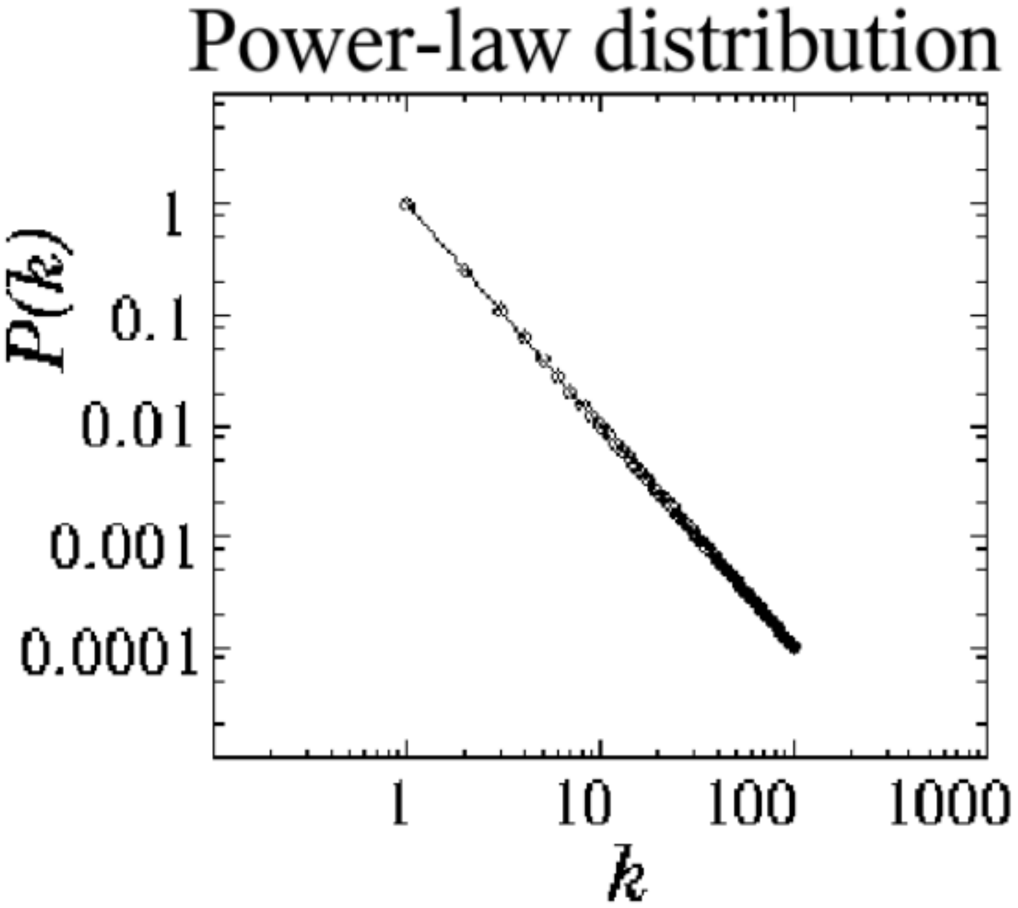


Small distance but large clustering (as in lattices)

# Network theory

## Models of networks.

## Barabási - Albert (random network with scale free property)



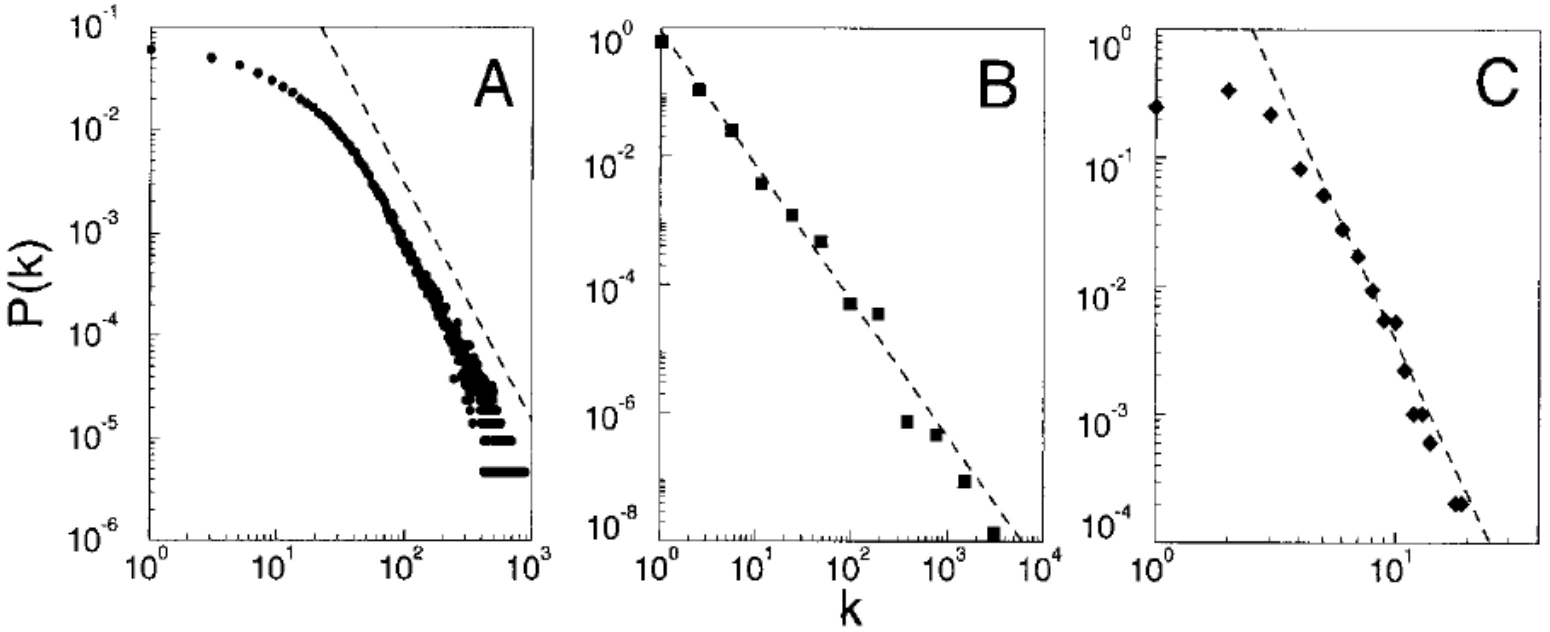


**Fig. 1.** The distribution function of connectivities for various large networks. (**A**) Actor collaboration graph with $N = 212{,}250$ vertices and average connectivity $\langle k \rangle = 28.78$. (**B**) WWW, $N = 325{,}729$, $\langle k \rangle = 5.46$ (6). (**C**) Power grid data, $N = 4941$, $\langle k \rangle = 2.67$. The dashed lines have slopes (A) $\gamma_{\mathrm{actor}} = 2.3$, (B) $\gamma_{\mathrm{www}} = 2.1$ and (C) $\gamma_{\mathrm{power}} = 4$.

# Network theory

Models of networks.

Barabási - Albert (random network with scale free property)

1) Start with m nodes connected among them

2) At each time step add 1 new node with a new link to existing nodes

3) Preferential attachement : the new node will select existing nodes according to their degree

$$P(\text{new node} \to i) = \frac{k_i}{\sum_j k_j}$$

Barabási, A.; Albert, R. (1999), "Emergence of scaling in random networks", *Science*, **286** (5439): 509–512.

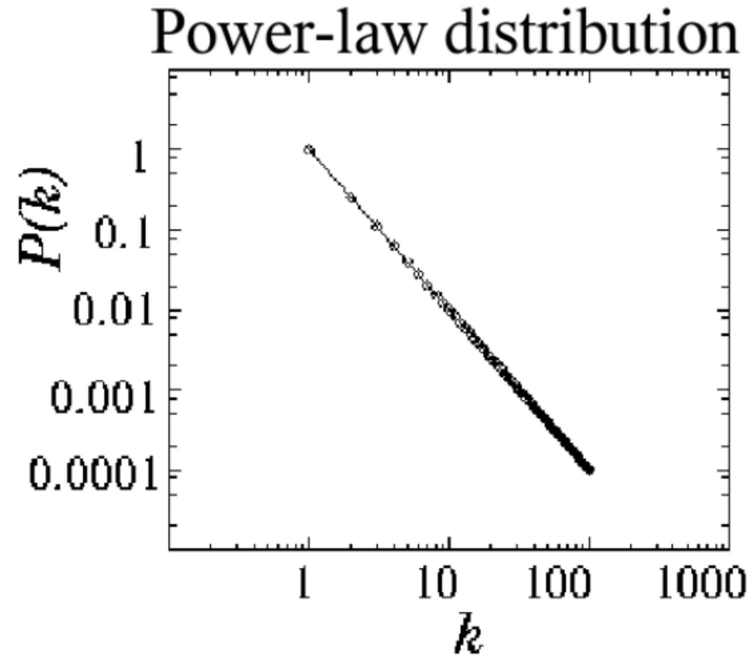
# Network theory

Models of networks.

Barabási - Albert (random network with scale free property)

In the limit of large networks one has

$$P(k) \sim \frac{1}{k^3}$$



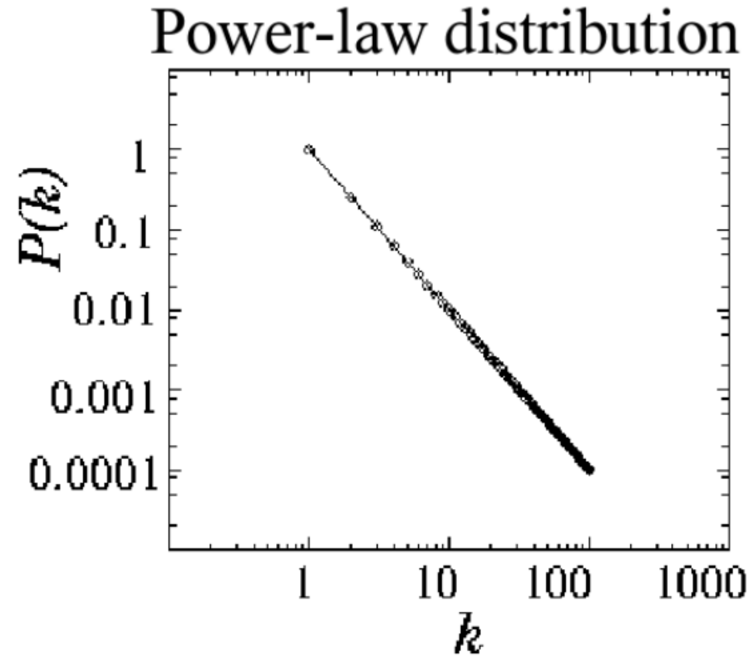
# Network theory

Models of networks.

Barabási - Albert (random network with scale free property)

In the limit of large networks one has

$$P(k) \sim \frac{1}{k^3}$$



A network for which $P(k) \sim \frac{1}{k^\gamma}$ is called scale free

## Models of networks.

Scale free network $P(k) \sim \frac{1}{k^\gamma}$

If $2 < \gamma \leq 3$ then $\langle k \rangle < \infty$ and $\langle k^2 \rangle = \infty$

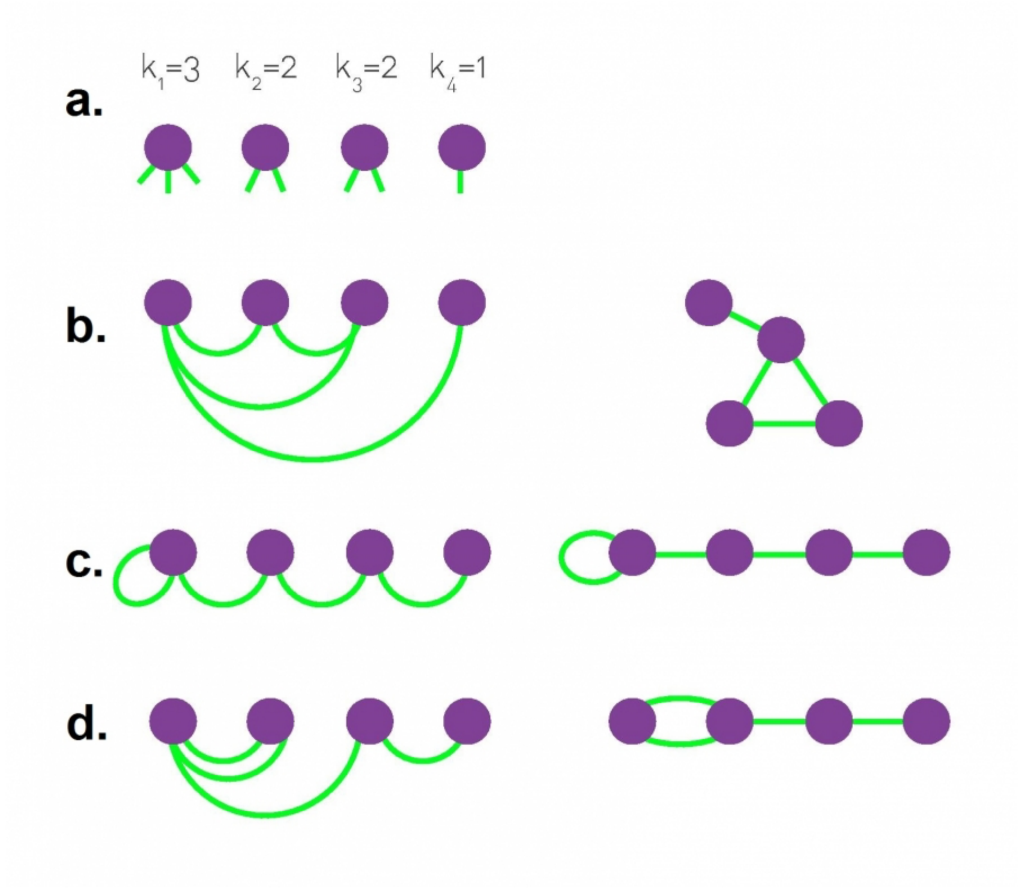
If $\gamma > 3$ then $\langle k \rangle < \infty$ and $\langle k^2 \rangle < \infty$

$$\langle k \rangle = \int_{k_{min}}^{k_{max}} k^{1-\gamma} dk \sim \int_{1}^{\infty} k^{1-\gamma} dk$$

Models of networks.

## Configuration model

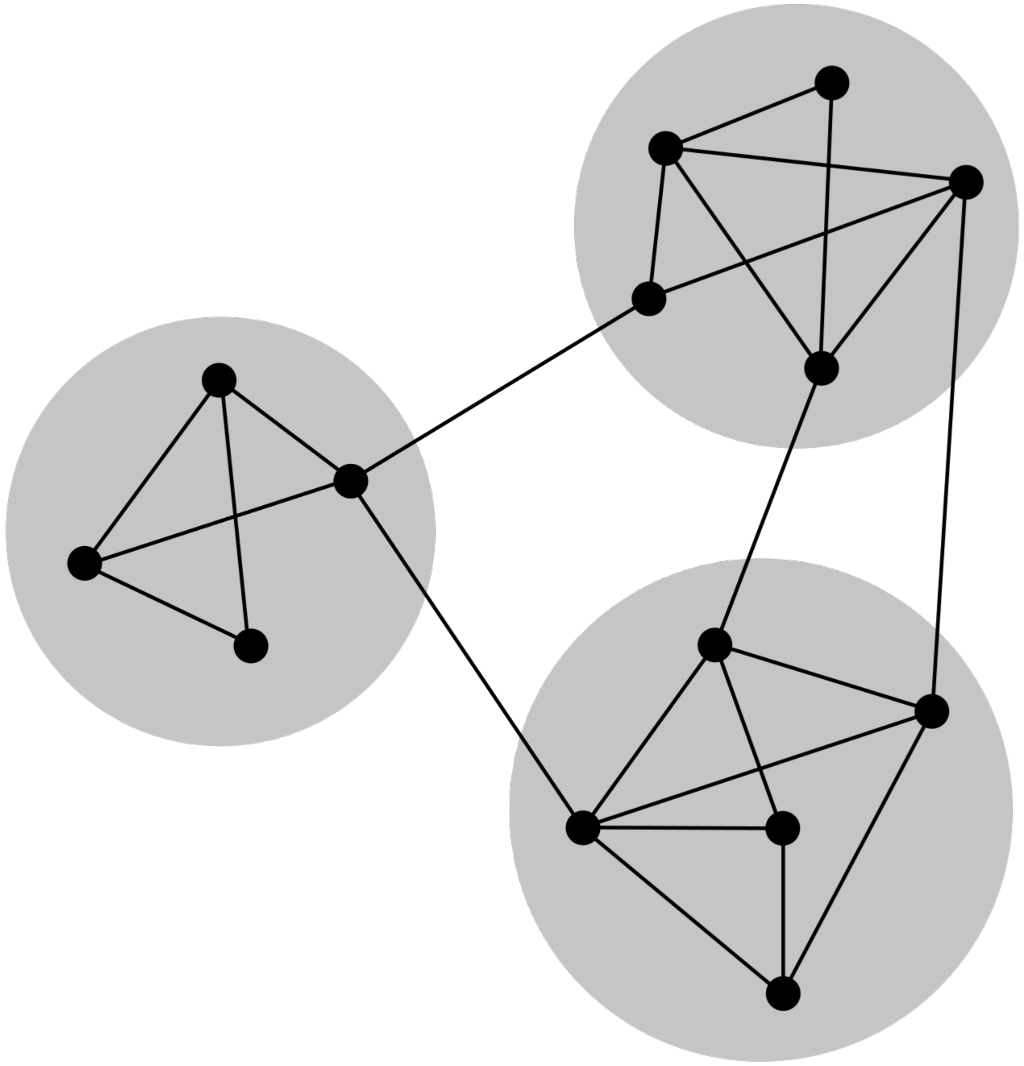
Given the degree sequence $k_1, \dots, k_n$ reconstruct the network that exhibits such degrees



# Network theory

## Community detection

Group of node tightly connected among them and weakly connected with the rest of the network.



# Network theory

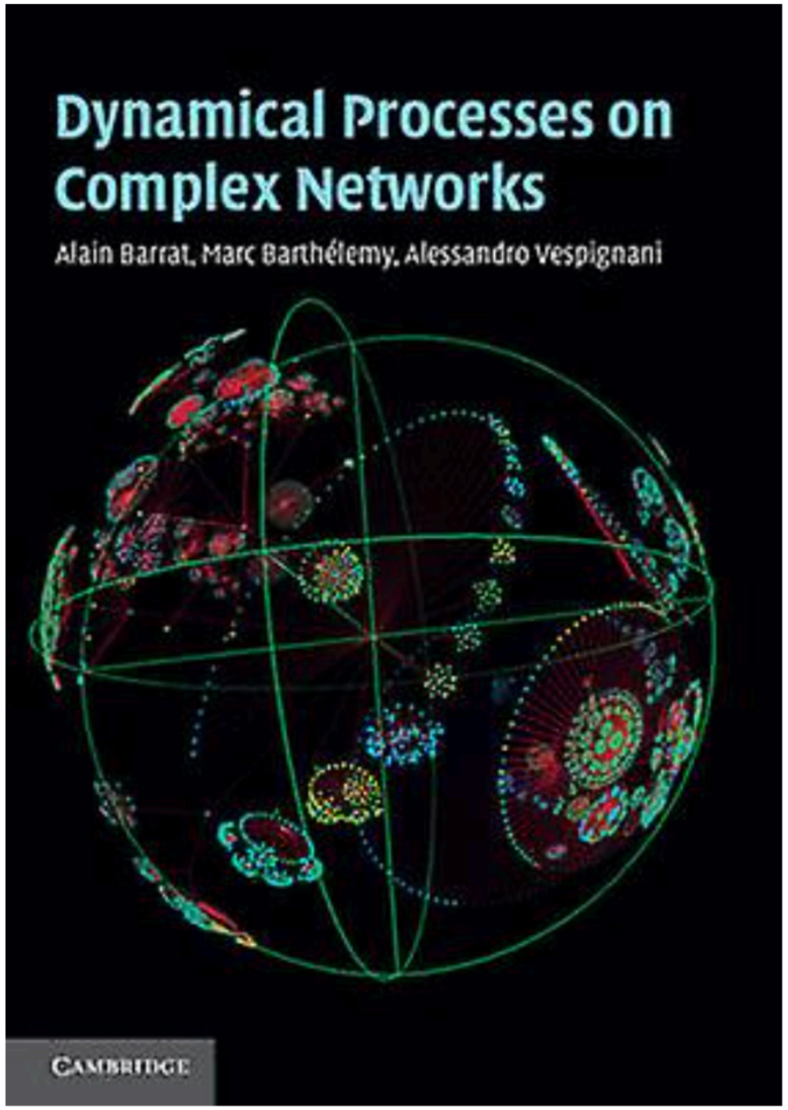
## Community detection

Group of node tightly connected among them and weakly connected with the rest of the network.

Communities allow to “simplify” the network structure

Communities allow to “better” understand the network dynamics

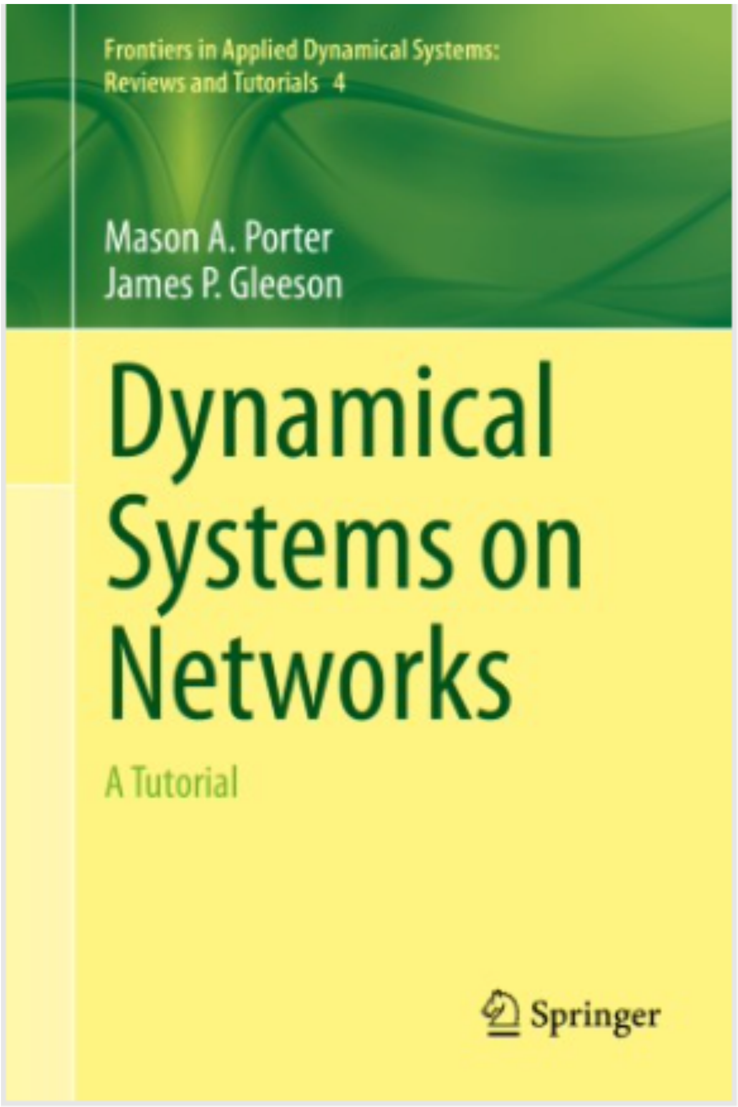
# Dynamical Systems on Network



Dynamical Processes on Complex Networks

Alain Barrat, Marc Barthélemy, Alessandro Vespignani

CAMBRIDGE



Frontiers in Applied Dynamical Systems: Reviews and Tutorials 4

Mason A. Porter
James P. Gleeson

Dynamical Systems on Networks

A Tutorial

Springer

# Dynamical Systems on Network

Assume to have n copies of the same dynamical system

$$\dot{\vec{x}}_i = \vec{f}(\vec{x}_i)$$

How to couple them?

What can be said about the global behaviour?

Are there some new emergent properties?

# Dynamical Systems on Network

Assume to have n copies of the same dynamical system

$$\dot{\vec{x}}_i = \vec{f}(\vec{x}_i)$$

How to couple them?

Coupling means interaction and interaction implies exchange.

Agents can thus move around the network (i.e., diffuse)

or they can stick on nodes and exchange “signals”.

# Dynamical Systems on Network

Assume to have n copies of the same dynamical system

$$\dot{\vec{x}}_i = \vec{f}(\vec{x}_i)$$

Agents can thus move around the network (i.e., diffuse)

Diffusive (like) coupling

or they can stick on nodes and exchange "signals".

Adjacency coupling

# Dynamical Systems on Network

Assume to have n copies of the same dynamical system

$$\dot{\vec{x}}_i = \vec{f}(\vec{x}_i)$$

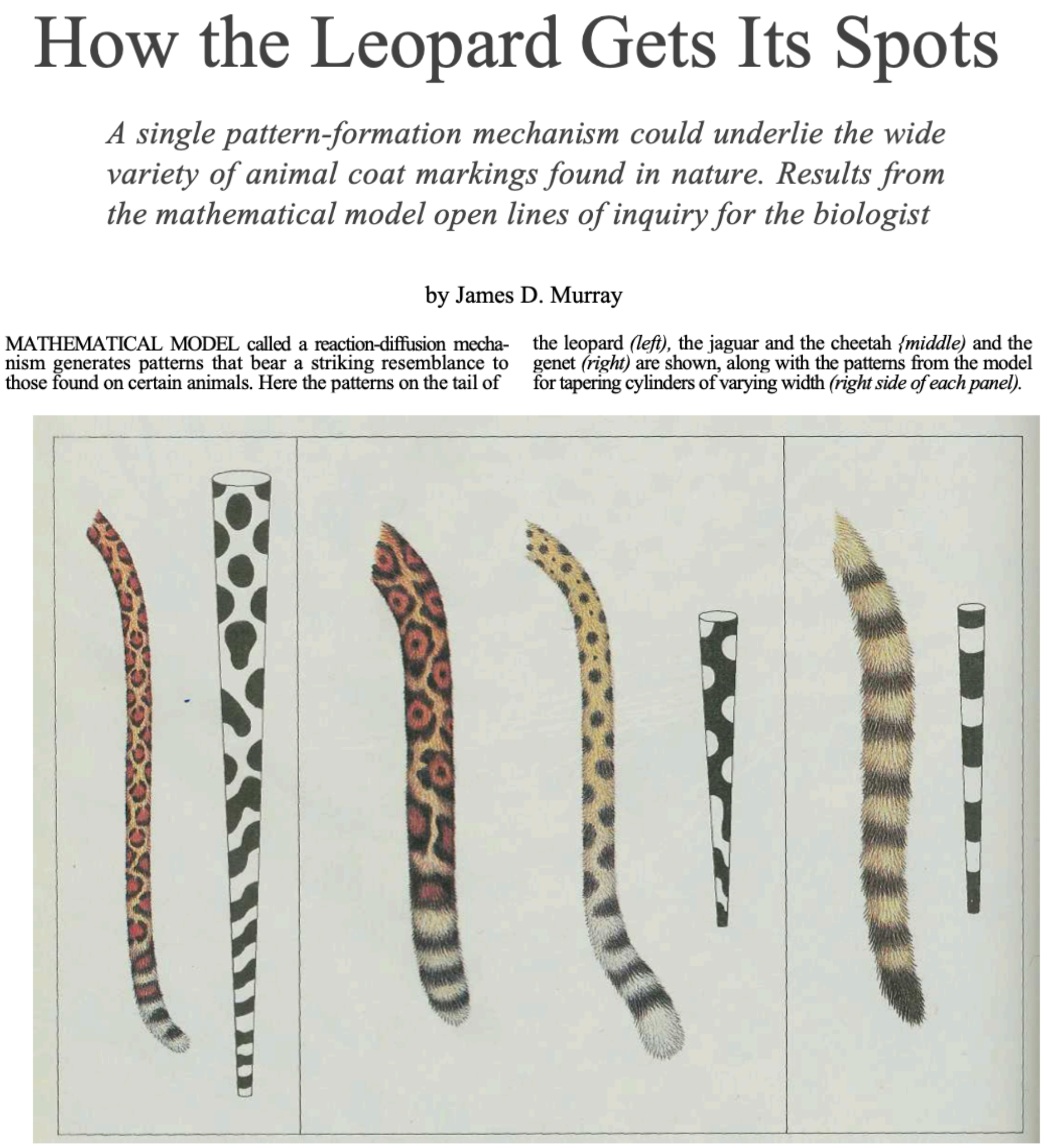
~~How to couple them?~~

What can be said about the global behaviour?

Are there some new emergent properties?

Assuming the isolated systems
do synchronise / converge to some equilibrium value,
does the coupled system do the same?

# Dynamical Systems on Network

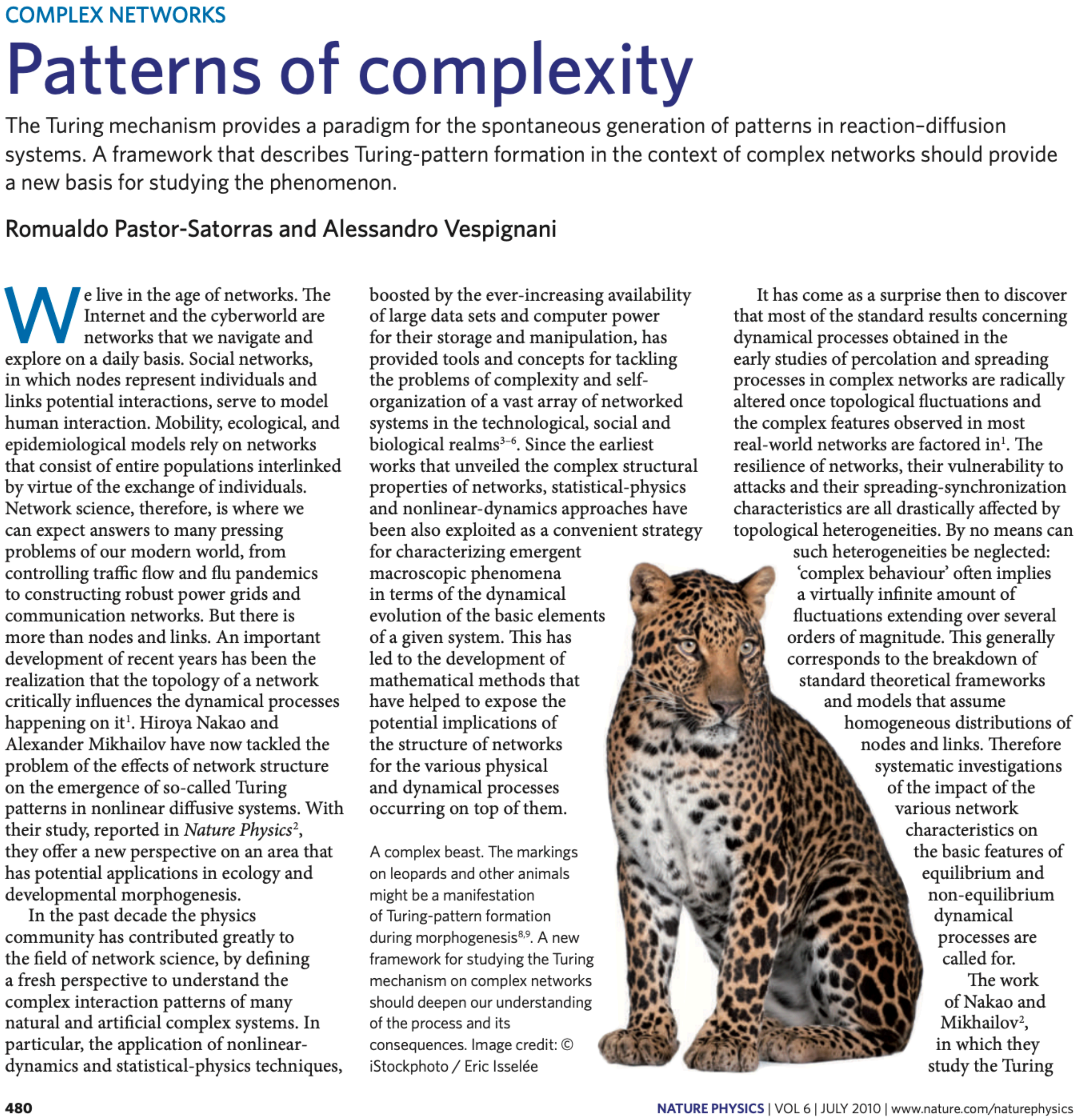


## How the Leopard Gets Its Spots

*A single pattern-formation mechanism could underlie the wide variety of animal coat markings found in nature. Results from the mathematical model open lines of inquiry for the biologist*

by James D. Murray

MATHEMATICAL MODEL called a reaction-diffusion mechanism generates patterns that bear a striking resemblance to those found on certain animals. Here the patterns on the tail of the leopard *(left)*, the jaguar and the cheetah *(middle)* and the genet *(right)* are shown, along with the patterns from the model for tapering cylinders of varying width *(right side of each panel)*.



COMPLEX NETWORKS

## Patterns of complexity

The Turing mechanism provides a paradigm for the spontaneous generation of patterns in reaction–diffusion systems. A framework that describes Turing-pattern formation in the context of complex networks should provide a new basis for studying the phenomenon.

**Romualdo Pastor-Satorras and Alessandro Vespignani**

We live in the age of networks. The Internet and the cyberworld are networks that we navigate and explore on a daily basis. Social networks, in which nodes represent individuals and links potential interactions, serve to model human interaction. Mobility, ecological, and epidemiological models rely on networks that consist of entire populations interlinked by virtue of the exchange of individuals. Network science, therefore, is where we can expect answers to many pressing problems of our modern world, from controlling traffic flow and flu pandemics to constructing robust power grids and communication networks. But there is more than nodes and links. An important development of recent years has been the realization that the topology of a network critically influences the dynamical processes happening on it[1]. Hiroya Nakao and Alexander Mikhailov have now tackled the problem of the effects of network structure on the emergence of so-called Turing patterns in nonlinear diffusive systems. With their study, reported in *Nature Physics*[2], they offer a new perspective on an area that has potential applications in ecology and developmental morphogenesis.

In the past decade the physics community has contributed greatly to the field of network science, by defining a fresh perspective to understand the complex interaction patterns of many natural and artificial complex systems. In particular, the application of nonlinear-dynamics and statistical-physics techniques, boosted by the ever-increasing availability of large data sets and computer power for their storage and manipulation, has provided tools and concepts for tackling the problems of complexity and self-organization of a vast array of networked systems in the technological, social and biological realms[3–6]. Since the earliest works that unveiled the complex structural properties of networks, statistical-physics and nonlinear-dynamics approaches have been also exploited as a convenient strategy for characterizing emergent macroscopic phenomena in terms of the dynamical evolution of the basic elements of a given system. This has led to the development of mathematical methods that have helped to expose the potential implications of the structure of networks for the various physical and dynamical processes occurring on top of them.

A complex beast. The markings on leopards and other animals might be a manifestation of Turing-pattern formation during morphogenesis[8,9]. A new framework for studying the Turing mechanism on complex networks should deepen our understanding of the process and its consequences. Image credit: © iStockphoto / Eric Isselée

It has come as a surprise then to discover that most of the standard results concerning dynamical processes obtained in the early studies of percolation and spreading processes in complex networks are radically altered once topological fluctuations and the complex features observed in most real-world networks are factored in[1]. The resilience of networks, their vulnerability to attacks and their spreading-synchronization characteristics are all drastically affected by topological heterogeneities. By no means can such heterogeneities be neglected: 'complex behaviour' often implies a virtually infinite amount of fluctuations extending over several orders of magnitude. This generally corresponds to the breakdown of standard theoretical frameworks and models that assume homogeneous distributions of nodes and links. Therefore systematic investigations of the impact of the various network characteristics on the basic features of equilibrium and non-equilibrium dynamical processes are called for.

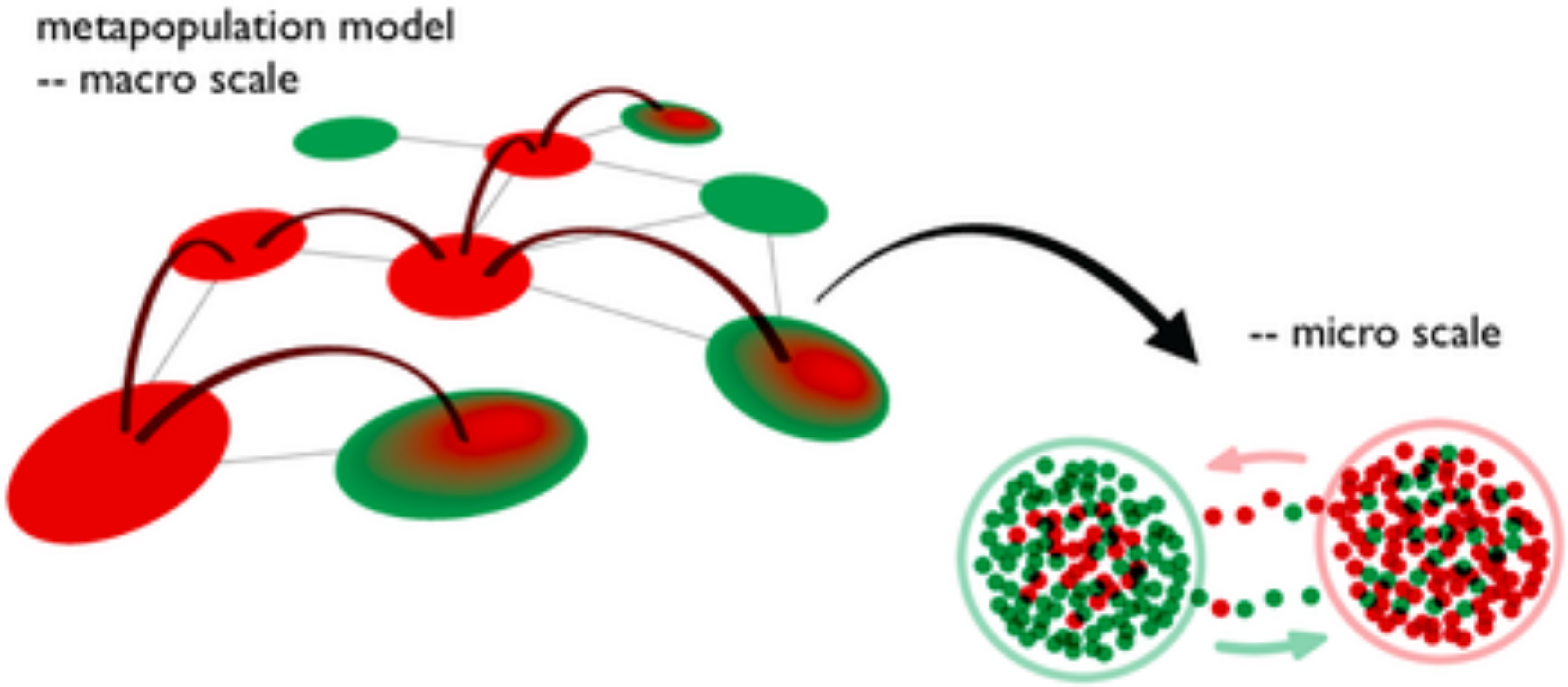
The work of Nakao and Mikhailov[2], in which they study the Turing

480

NATURE PHYSICS | VOL 6 | JULY 2010 | www.nature.com/naturephysics

# Diffusive coupling

# Dynamical Systems on Network



## Metapopulation models

e.g. in the framework of ecology:
May R., *Will a large complex system be stable?* Nature, 238, pp. 413, (1972)

Interactions occur at each node. Diffusion occurs across edges.

# Dynamical Systems on Network

Assume each isolated system converges to the same stable stationary equilibrium

$$\lim_{t\to\infty} \vec{x}_i(t) = \vec{x}_i^* \quad \forall i$$

Assume a diffusive coupling

$$\dot{\vec{x}}_i \sim \sum_j A_{ij}(\vec{x}_j - \vec{x}_i)$$

# Dynamical Systems on Network

Assume each isolated system converges to the same stable stationary equilibrium

$$\lim_{t\to\infty}\vec{x}_i(t)=\vec{x}_i^*\quad\forall i$$

Assume a diffusive coupling

$$\dot{\vec{x}}_i\sim\sum_j A_{ij}(\vec{x}_j-\vec{x}_i)\;=\sum_j A_{ij}\vec{x}_j-k_i\vec{x}_i=$$

$$=\sum_j\left(A_{ij}-k_i\delta_{ij}\right)\vec{x}_j=:\sum_j L_{ij}\vec{x}_j$$

$\mathbf{L}=\mathbf{A}-\mathrm{diag}(k_1,\dots,k_n)$ is the (combinatorial) Laplace matrix

# Dynamical Systems on Network

Some properties of $\mathbf{L} = \mathbf{A} - \mathrm{diag}(k_1, \dots, k_n)$

$L_{ij} = L_{ji} \;\Rightarrow$ Diagonalisable and with real eigenvalues

# Dynamical Systems on Network

Some properties of $\mathbf{L} = \mathbf{A} - \mathrm{diag}(k_1, \dots, k_n)$

$L_{ij} = L_{ji} \;\Rightarrow$ Diagonalisable and with real eigenvalues

$$\sum_j L_{ij} = 0 \quad \forall i \;\Rightarrow\; \Lambda^{(1)} = 0 \quad \vec{\phi}^{(1)} = (1, \dots, 1)^\top / \sqrt{n}$$

## Dynamical Systems on Network

Some properties of $\mathbf{L} = \mathbf{A} - \mathrm{diag}(k_1, \dots, k_n)$

$L_{ij} = L_{ji} \Rightarrow$ Diagonalisable and with real eigenvalues

$$\sum_j L_{ij} = 0 \quad \forall i \quad \Rightarrow \quad \Lambda^{(1)} = 0 \quad \vec{\phi}^{(1)} = (1, \dots, 1)^\top / \sqrt{n}$$

Non-positive definite $-\mathbf{M}^\top \mathbf{M} = \mathbf{A} - \mathrm{diag}(k_1, k_2, k_3) = \mathbf{L}$

$$\begin{aligned}(\phi^{(\alpha)}, \mathbf{L}\phi^{(\alpha)}) &= -(\phi^{(\alpha)}, \mathbf{M}^\top \mathbf{M}\phi^{(\alpha)}) = \\ &= -(\mathbf{M}^\top \phi^{(\alpha)}, \mathbf{M}\phi^{(\alpha)}) = -||\mathbf{M}\phi^{(\alpha)}||^2 \leq 0\end{aligned}$$

# Dynamical Systems on Network

$$\dot{\vec{x}}_i = \vec{f}(\vec{x}_i) + \mathbf{D}\sum_j L_{ij}\vec{x}_j \quad \forall i$$

Diffusion coefficients $\mathbf{D}$

Under which conditions on the network, i.e., on $\mathbf{L}$, we can destabilise the equilibrium $\vec{x}^*$ ?

Let us observe that $\vec{x}^*$ is also a solution of the coupled systems

because $$\sum_j L_{ij}\vec{x}^*_j = 0 \quad \forall i$$

## Dynamical Systems on Network

$$\dot{\vec{x}}_i = \vec{f}(\vec{x}_i) + \mathbf{D} \sum_j L_{ij} \vec{x}_j \quad \forall i$$ Diffusion coefficients $\mathbf{D}$

Under which conditions on the network, i.e., on $\mathbf{L}$, we can destabilise the equilibrium $\vec{x}^*$ ?

Let us observe that $\vec{x}^*$ is also a solution of the coupled systems

because $$\sum_j L_{ij} \vec{x}^*_j = 0 \quad \forall i$$

This is the celebrated Turing instability phenomenon

# Dynamical Systems on Network

We have to show that

$\vec{x}^*$ is stable for $\dot{\vec{x}}_i = \vec{f}(\vec{x}_i) \quad \forall i$ (decoupled systems)

$\vec{x}^*$ is unstable for $\dot{\vec{x}}_i = \vec{f}(\vec{x}_i) + \mathbf{D}\sum_j L_{ij}\vec{x}_j \quad \forall i$ (coupled systems)

under suitable assumptions on $\mathbf{L}$ and $\mathbf{D}$

Following the instability, the system will (possibly) reach a new equilibrium, in general not homogeneous, i.e., the patchy solution.

Nakao, H, Mikhailov, AS, 2010 Turing patterns in network-organized activator–inhibitor systems. Nature Physics, 6, 544

# Dynamical Systems on Network

## A relevant example : the Brusselator

$$\begin{cases} \dot{u}_i & = 1 - (b+1)u_i + cu_i^2 v_i + D_u \sum_j L_{ij} u_j \\ \dot{v}_i & = bu_i - cu_i^2 v_i + D_v \sum_j L_{ij} v_j \end{cases} \qquad \forall i$$

$(u^*, v^*) = (1, b/c)$ equilibrium isolated system

## Dynamical Systems on Network

A relevant example : the Brusselator

$$\begin{cases} \dot{u}_i &= 1-(b+1)u_i + cu_i^2 v_i + D_u \sum_j L_{ij} u_j \\ \dot{v}_i &= bu_i - cu_i^2 v_i + D_v \sum_j L_{ij} v_j \end{cases} \qquad \forall i$$

$(u^*, v^*) = (1, b/c)$ equilibrium isolated system

$$x_i = u_i - u^* \, , y_i = v_i - v^*$$

$$\begin{cases} \dot{x}_i &= (b-1)x_i + cy_i + D_u \sum_j L_{ij} x_j \\ \dot{y}_i &= -bx_i - cy_i + D_v \sum_j L_{ij} y_j \end{cases}$$

linearised system

# Dynamical Systems on Network

The stability of $(u^*, v^*) = (1, b/c)$ can be stated by looking at

the spectrum of $\mathbf{J}_0 = \begin{pmatrix} b-1 & c \\ -b & -c \end{pmatrix}$

$$\lambda^2 - \lambda(b-1-c) + c = 0$$

$$\lambda = \frac{(b-1-c) \pm \sqrt{(b-1-c)^2 - 4c}}{2}$$

# Dynamical Systems on Network

The stability of $(u^*, v^*) = (1, b/c)$ can be stated by looking at the spectrum of

$$\mathbf{J}_0 = \begin{pmatrix} b-1 & c \\ -b & -c \end{pmatrix}$$

$$\lambda^2 - \lambda(b-1-c) + c = 0$$

$$\lambda = \frac{(b-1-c) \pm \sqrt{(b-1-c)^2 - 4c}}{2}$$

Note

$$\mathbf{J}_0 = \begin{pmatrix} \partial_u f & \partial_v f \\ \partial_u g & \partial_v g \end{pmatrix} \qquad \lambda^2 - \lambda \mathrm{tr}(\mathbf{J}_0) + \det(\mathbf{J}_0) = 0$$

$\mathrm{tr}(\mathbf{J}_0) < 0$ and $\det(\mathbf{J}_0) > 0$ imply stability

# Dynamical Systems on Network

$$\frac{d}{dt}\begin{pmatrix} x_i \\ y_i \end{pmatrix} = \begin{pmatrix} b-1 & c \\ -b & -c \end{pmatrix}\begin{pmatrix} x_i \\ y_i \end{pmatrix} + \sum_j L_{ij}\begin{pmatrix} D_u & 0 \\ 0 & D_v \end{pmatrix}\begin{pmatrix} x_i \\ y_i \end{pmatrix}$$

Let us decompose x and y on the eigenbasis of L

$$x_i = \sum_\alpha \xi_\alpha \phi_i^{(\alpha)} \quad \text{and} \quad y_i = \sum_\alpha \eta_\alpha \phi_i^{(\alpha)}$$

$$\sum_j L_{ij} x_j = \sum_\alpha \sum_j L_{ij} \xi_\alpha \phi_j^{(\alpha)} = \sum_\alpha \Lambda^{(\alpha)} \xi_\alpha \phi_i^{(\alpha)}$$

# Dynamical Systems on Network

$$\frac{d}{dt}\begin{pmatrix} x_i \\ y_i \end{pmatrix} = \begin{pmatrix} b-1 & c \\ -b & -c \end{pmatrix}\begin{pmatrix} x_i \\ y_i \end{pmatrix} + \sum_j L_{ij}\begin{pmatrix} D_u & 0 \\ 0 & D_v \end{pmatrix}\begin{pmatrix} x_i \\ y_i \end{pmatrix}$$

Let us decompose x and y on the eigenbasis of L

$$x_i = \sum_\alpha \xi_\alpha \phi_i^{(\alpha)} \quad \text{and} \quad y_i = \sum_\alpha \eta_\alpha \phi_i^{(\alpha)}$$

$$\sum_j L_{ij} x_j = \sum_\alpha \sum_j L_{ij} \xi_\alpha \phi_j^{(\alpha)} = \sum_\alpha \Lambda^{(\alpha)} \xi_\alpha \phi_i^{(\alpha)}$$

Project on each eigenvector

$$\frac{d}{dt}\begin{pmatrix} \xi_\alpha \\ \eta_\alpha \end{pmatrix} = \begin{pmatrix} b-1 & c \\ -b & -c \end{pmatrix}\begin{pmatrix} \xi_\alpha \\ \eta_\alpha \end{pmatrix} + \Lambda^{(\alpha)}\begin{pmatrix} D_u & 0 \\ 0 & D_v \end{pmatrix}\begin{pmatrix} \xi_\alpha \\ \eta_\alpha \end{pmatrix}$$

# Dynamical Systems on Network

$$\mathbf{J}_\alpha := \begin{pmatrix} b-1 & c \\ -b & -c \end{pmatrix} + \Lambda^{(\alpha)} \begin{pmatrix} D_u & 0 \\ 0 & D_v \end{pmatrix} = \mathbf{J}_0 + \Lambda^{(\alpha)} \mathbf{D}$$

Growth ansatz $\xi_\alpha(t) \sim e^{\lambda_\alpha t}$ and $\eta_\alpha(t) \sim e^{\lambda_\alpha t}$ $\lambda_\alpha = \lambda_\alpha(\Lambda^{(\alpha)})$

$$\det \left[ \begin{pmatrix} b-1 & c \\ -b & -c \end{pmatrix} + \Lambda^{(\alpha)} \begin{pmatrix} D_u & 0 \\ 0 & D_v \end{pmatrix} - \lambda_\alpha \mathbb{I} \right] = 0$$

# Dynamical Systems on Network

$$\mathbf{J}_\alpha := \begin{pmatrix} b-1 & c \\ -b & -c \end{pmatrix} + \Lambda^{(\alpha)} \begin{pmatrix} D_u & 0 \\ 0 & D_v \end{pmatrix} = \mathbf{J}_0 + \Lambda^{(\alpha)}\mathbf{D}$$

Growth ansatz $\xi_\alpha(t) \sim e^{\lambda_\alpha t}$ and $\eta_\alpha(t) \sim e^{\lambda_\alpha t}$ $\lambda_\alpha = \lambda_\alpha(\Lambda^{(\alpha)})$

$$\det\left[\begin{pmatrix} b-1 & c \\ -b & -c \end{pmatrix} + \Lambda^{(\alpha)} \begin{pmatrix} D_u & 0 \\ 0 & D_v \end{pmatrix} - \lambda_\alpha \mathbb{I}\right] = 0$$

$$\lambda_\alpha^2 - \lambda_\alpha \mathrm{tr}(\mathbf{J}_\alpha) + \det(\mathbf{J}_\alpha) = 0$$

instability $\mathrm{tr}(\mathbf{J}_\alpha) \geq 0$ or $\det(\mathbf{J}_\alpha) \leq 0$ for some $\alpha$

but $\mathrm{tr}(\mathbf{J}_\alpha) = \mathrm{tr}(\mathbf{J}_0) + \Lambda^{(\alpha)}(D_u + D_v) \leq \mathrm{tr}(\mathbf{J}_0) < 0$

# Dynamical Systems on Network

$\exists\alpha \text{ st } \det \mathbf{J}_\alpha \leq 0$ instability

# Dynamical Systems on Network

$\exists \alpha$ st $\det \mathbf{J}_\alpha \leq 0$ instability

$$\det(\mathbf{J}_\alpha) = \underbrace{D_u D_v \left[\Lambda^{(\alpha)}\right]^2}_{\text{positive}} + \underbrace{\Lambda^{(\alpha)}}_{\text{negative}} \left(D_v \partial_u f + D_u \partial_v g\right) + \underbrace{\det(\mathbf{J}_0)}_{\text{positive}}$$

$$D_v \partial_u f + D_u \partial_v g > 0$$

# Dynamical Systems on Network

$\exists\alpha \text{ st } \det \mathbf{J}_\alpha \leq 0$ instability

$$\det(\mathbf{J}_\alpha) = \underbrace{D_u D_v \left[\Lambda^{(\alpha)}\right]^2}_{\text{positive}} + \underbrace{\Lambda^{(\alpha)}}_{\text{negative}} \left(D_v \partial_u f + D_u \partial_v g\right) + \underbrace{\det(\mathbf{J}_0)}_{\text{positive}}$$

$$\left.\begin{array}{l} D_v \partial_u f + D_u \partial_v g > 0 \\ \text{but} \\ \mathrm{tr}(\mathbf{J}_0) = \partial_u f + \partial_v g < 0 \end{array}\right\}$$

$\partial_u f$ and $\partial_v g$ should have opposite sign, i.e., should be activator and inhibitor

$\partial_u f > 0 \qquad \partial_v g < 0$

# Dynamical Systems on Network

Note : the inhibitor should diffuse faster than the activator

$$D_v > D_u$$

Gierer A, Meinhardt H. 1972 A theory of biological pattern formation. Kybernetik 12, 30

# Dynamical Systems on Network

$\exists\alpha$ st $\det \mathbf{J}_\alpha \leq 0$ instability

$$\det(\mathbf{J}_\alpha) = D_u D_v \left[\Lambda^{(\alpha)}\right]^2 + \Lambda^{(\alpha)} \left(D_v \partial_u f + D_u \partial_v g\right) + \det(\mathbf{J}_0)$$

$$-\frac{1}{4}\frac{\left(D_v \partial_u f + D_u \partial_v g\right)^2}{D_u D_v} + \det(\mathbf{J}_0) < 0$$

# Turing conditions

$$\mathrm{tr}(\mathbf{J}_0) = \partial_u f + \partial_v g < 0$$

$$\det(\mathbf{J}_0) = \partial_u f \partial_v g - \partial_v f \partial_u g > 0$$

$$D_v \partial_u f + D_u \partial_v g > 0$$

$$-\frac{1}{4}\frac{(D_v \partial_u f + D_u \partial_v g)^2}{D_u D_v} + \det(\mathbf{J}_0) < 0$$

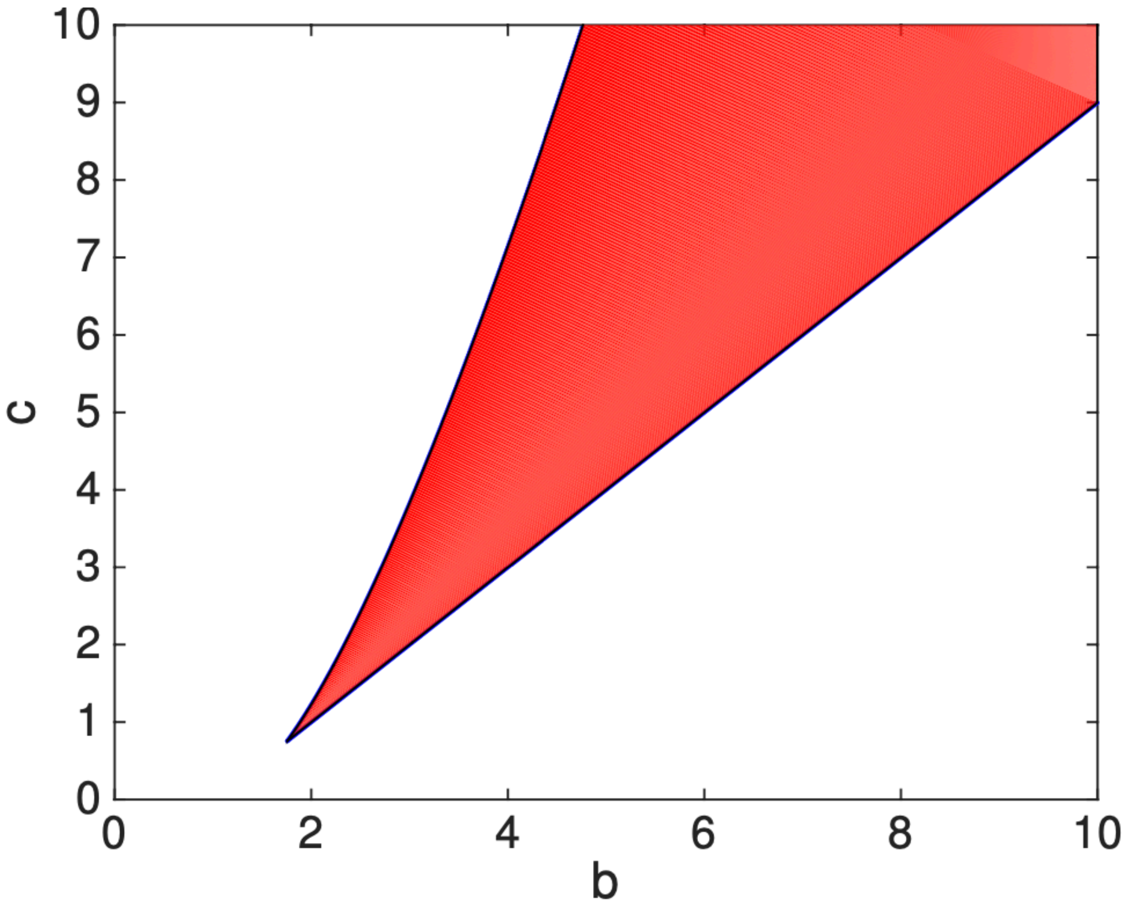
Turing AM. 1952 The chemical basis of morphogenesis. Phil. Trans. R. Soc. Lond. B 237, 37

# Turing conditions for the Brusselator model

$$\mathrm{tr}(\mathbf{J}_0) = b - 1 - c < 0$$

$$\det(\mathbf{J}_0) = c > 0$$

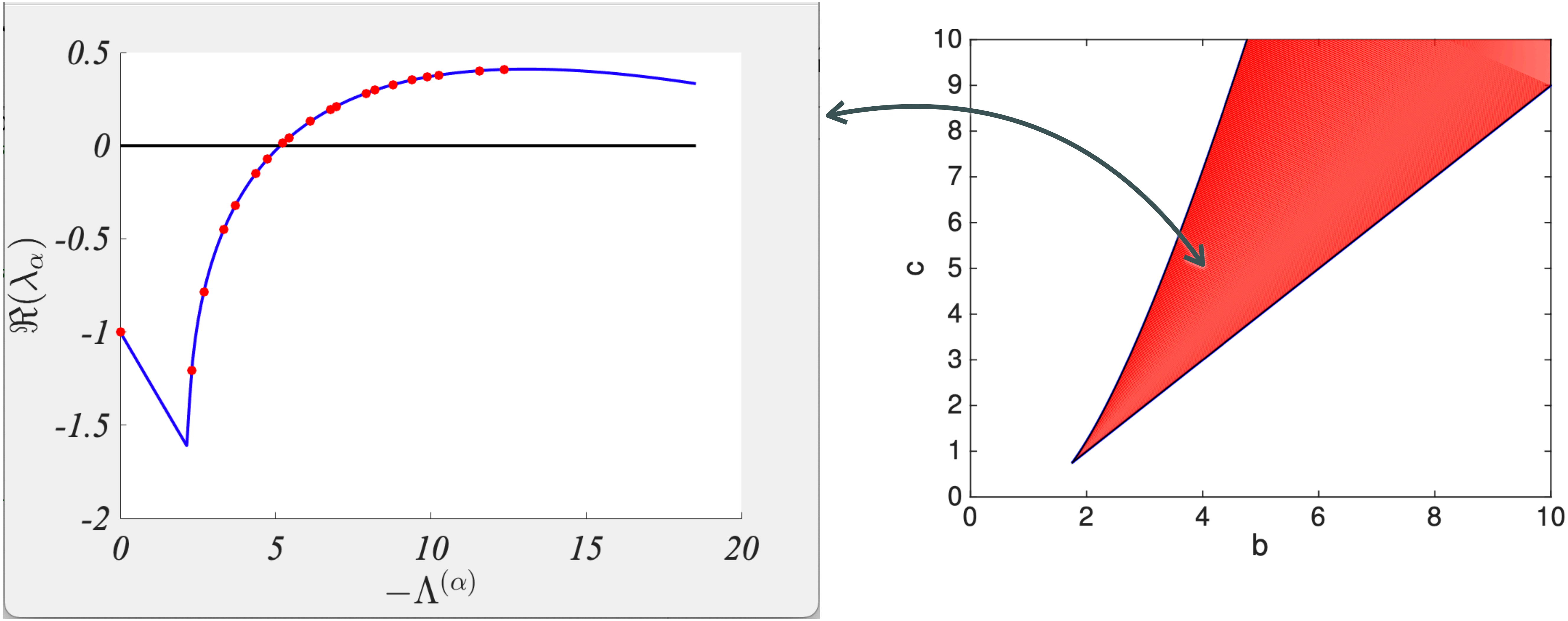
$$D_v(b-1) - D_u c > 0$$



$$c < \frac{D_v}{D_u}(b+1)\left[1 - \sqrt{1 - \frac{(b-1)^2}{(b+1)^2}}\right]$$

# Dynamical Systems on Network

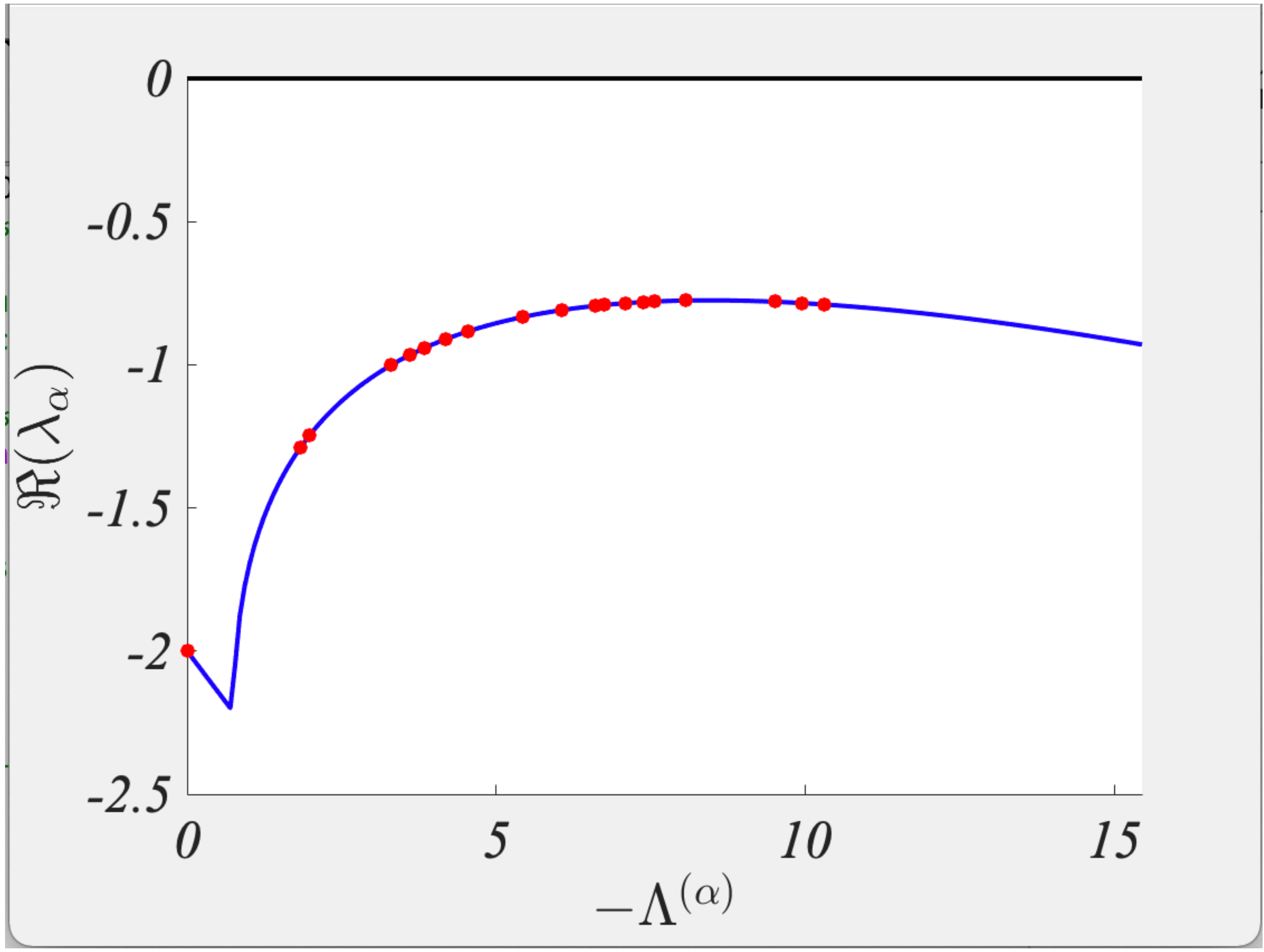
Growth ansatz $\xi_\alpha(t) \sim e^{\lambda_\alpha t}$ and $\eta_\alpha(t) \sim e^{\lambda_\alpha t}$ $\lambda_\alpha = \lambda_\alpha(\Lambda^{(\alpha)})$

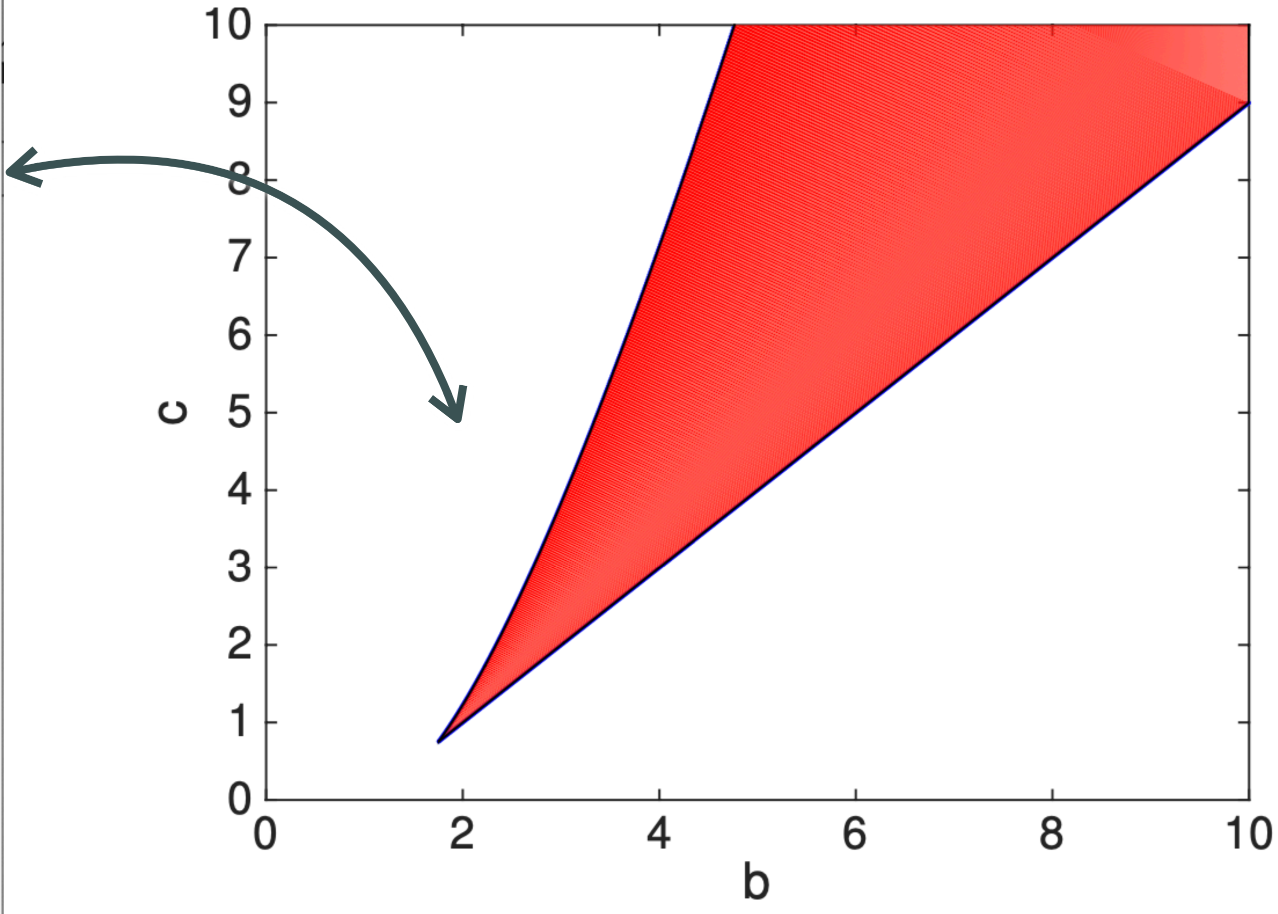


$$\boxed{b = 4}, c = 5\,, D_u = 0.07\,, D_v = 0.5$$

# Dynamical Systems on Network

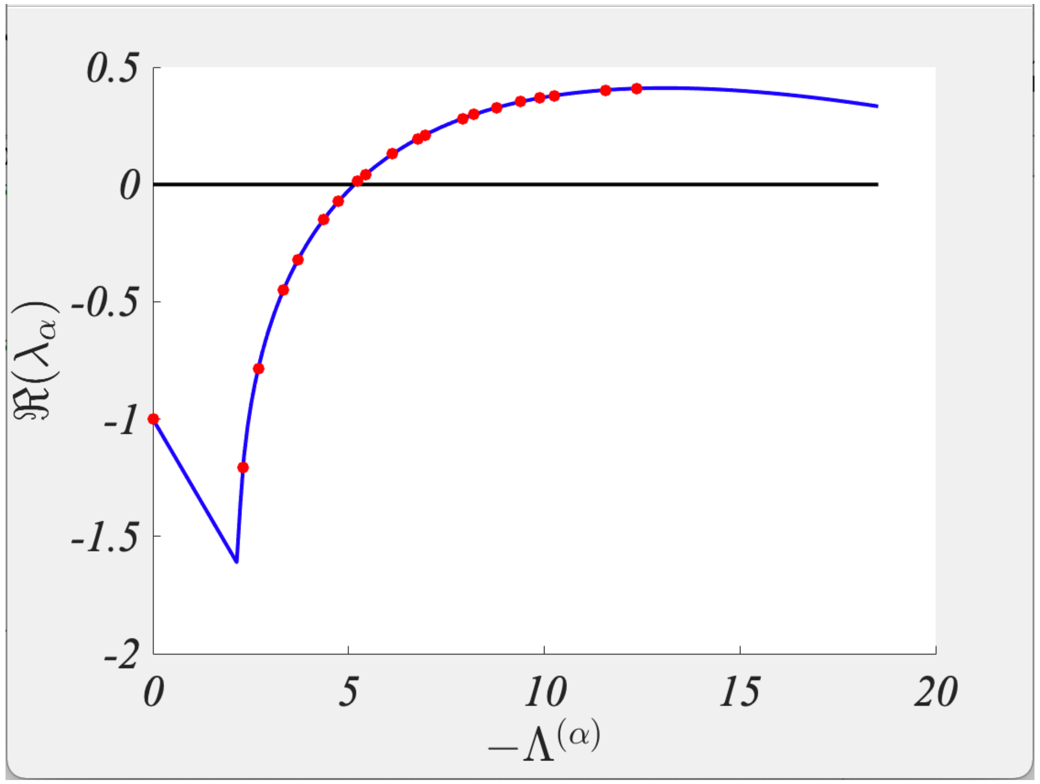
Growth ansatz $\xi_\alpha(t) \sim e^{\lambda_\alpha t}$ and $\eta_\alpha(t) \sim e^{\lambda_\alpha t}$ $\lambda_\alpha = \lambda_\alpha(\Lambda^{(\alpha)})$

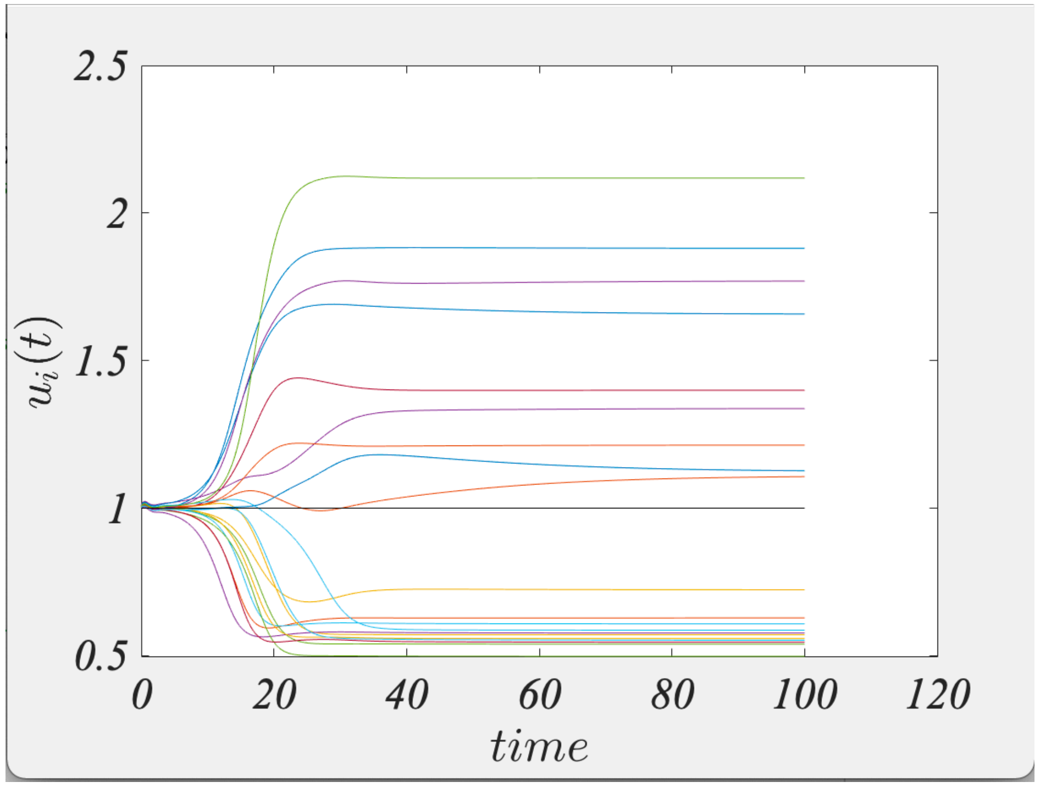




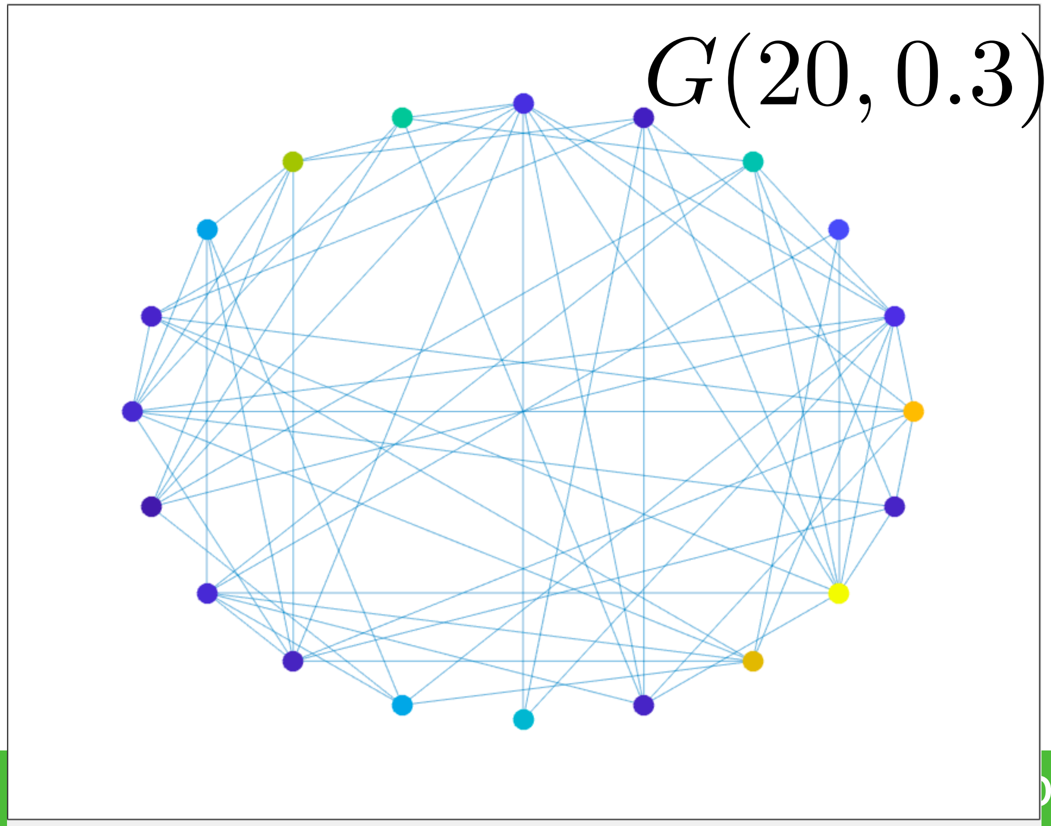
$$\boxed{b = 2}\,, c = 5\,, D_u = 0.07\,, D_v = 0.5$$

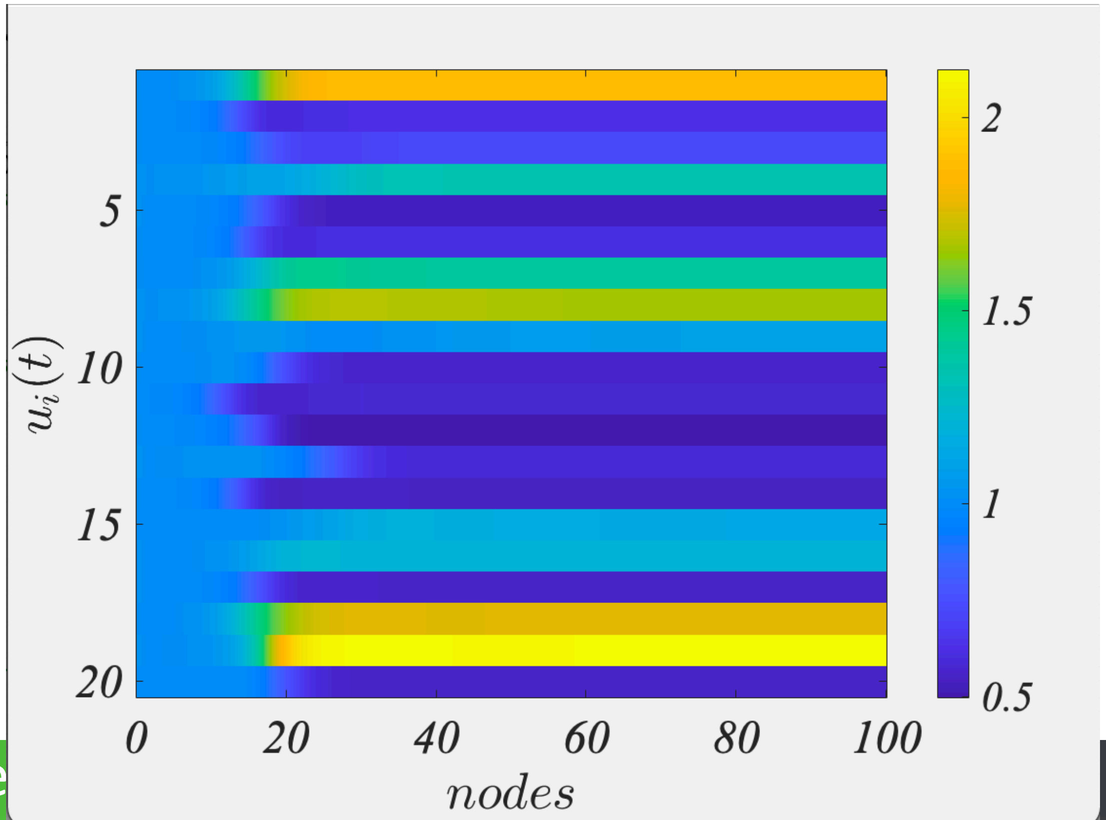
# Dynamical Systems on Network



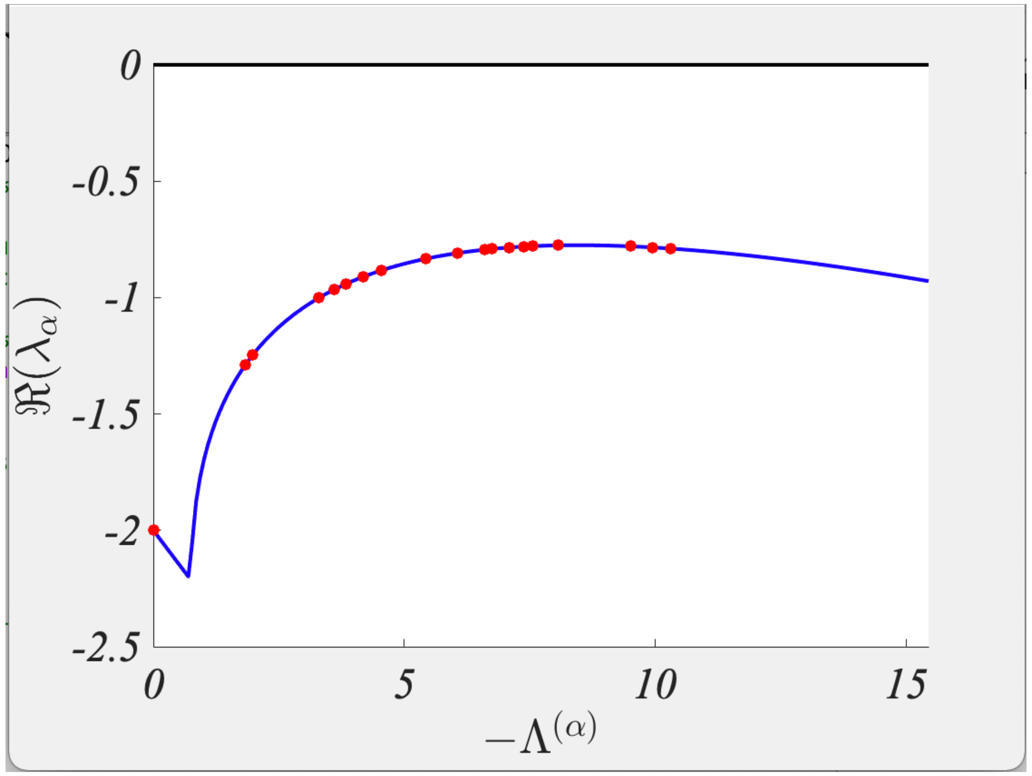


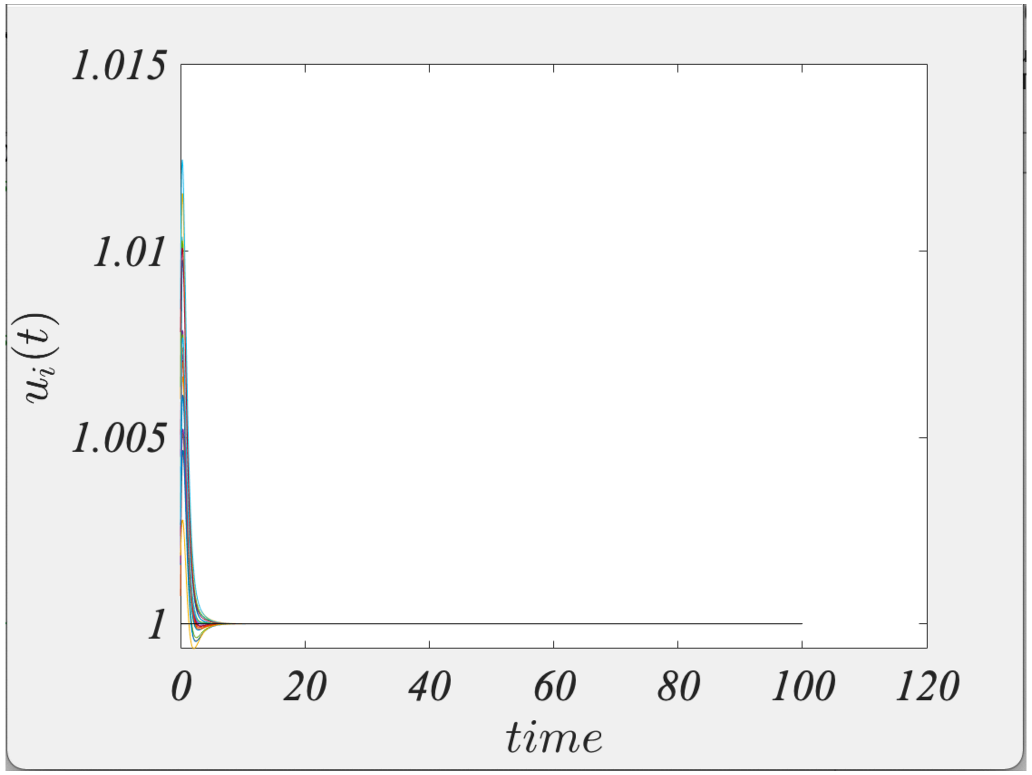
Erdős–Rényi (random network)





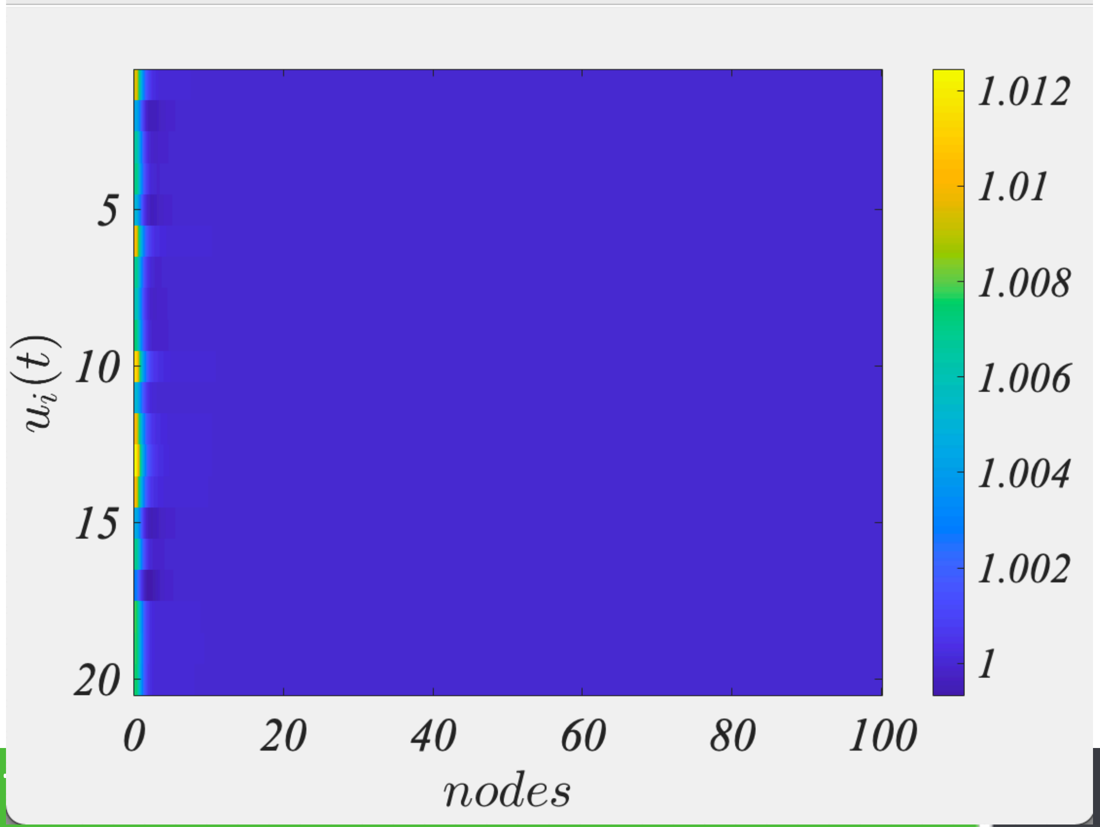
# Dynamical Systems on Network





Erdős–Rényi (random network)

$$G(20, 0.3)$$



# Directed networks

Assume a diffusive coupling

$$\dot{\vec{x}}_i \sim \sum_j A_{ij}(\vec{x}_j - \vec{x}_i) = \sum_j A_{ij}\vec{x}_j - k_i^{(in)}\vec{x}_i$$

$$= \sum_j (A_{ij}\vec{x}_j - k_i^{(in)}\delta_{ij})\vec{x}_j =: \sum_j L_{ij}\vec{x}_j$$

$$\mathbf{L} = \mathbf{A} - \mathrm{diag}(k_1^{(in)}, \dots, k_n^{(in)})$$

is the (combinatorial) directed Laplace matrix

It is not symmetric, thus the eigenvalues can be complex numbers

# Directed networks

stability of the homogeneous equilibrium decoupled systems

$$\lambda^2 - \lambda \mathrm{tr}(\mathbf{J}_0) + \det(\mathbf{J}_0) = 0$$

$$\mathrm{tr}(\mathbf{J}_0) < 0 \text{ and } \det(\mathbf{J}_0) > 0$$

instability of the homogeneous equilibrium coupled systems

$$\lambda_\alpha^2 - \lambda_\alpha \mathrm{tr}(\mathbf{J}_\alpha) + \det(\mathbf{J}_\alpha) = 0$$

$$\mathrm{tr}(\mathbf{J}_\alpha) \geq 0 \text{ or } \det(\mathbf{J}_\alpha) \leq 0$$ for some $\alpha$

Asllani M et al. 2014 The theory of pattern formation on directed networks. Nature Communication 5, 4517

# Directed networks

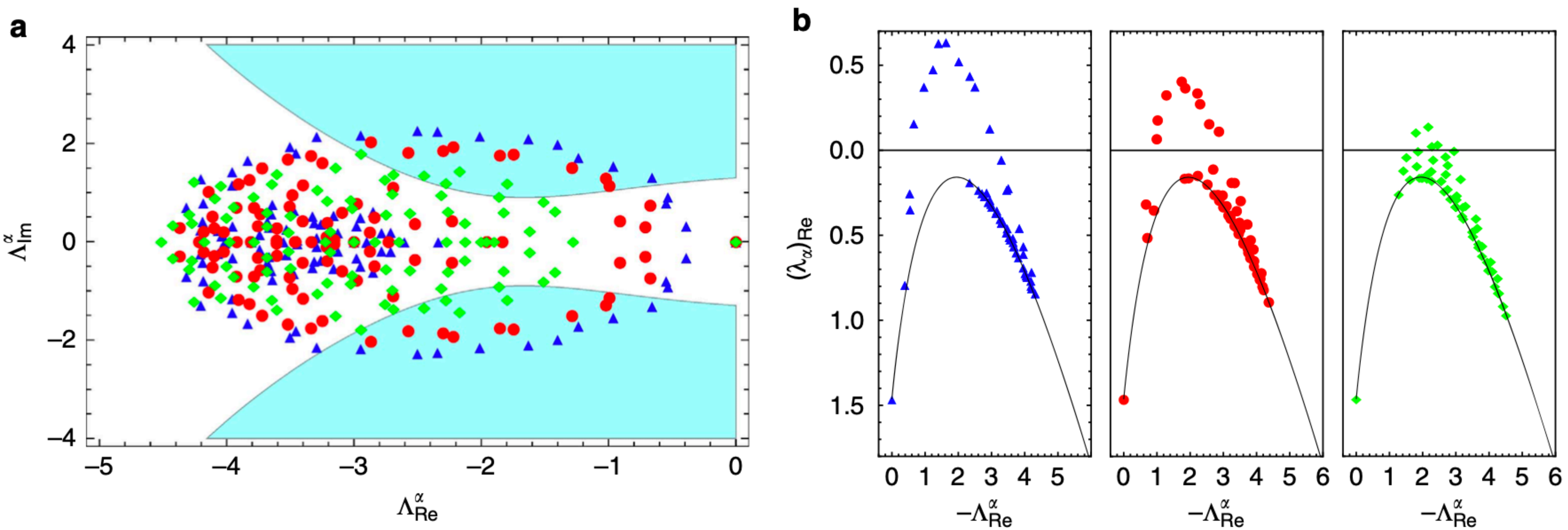
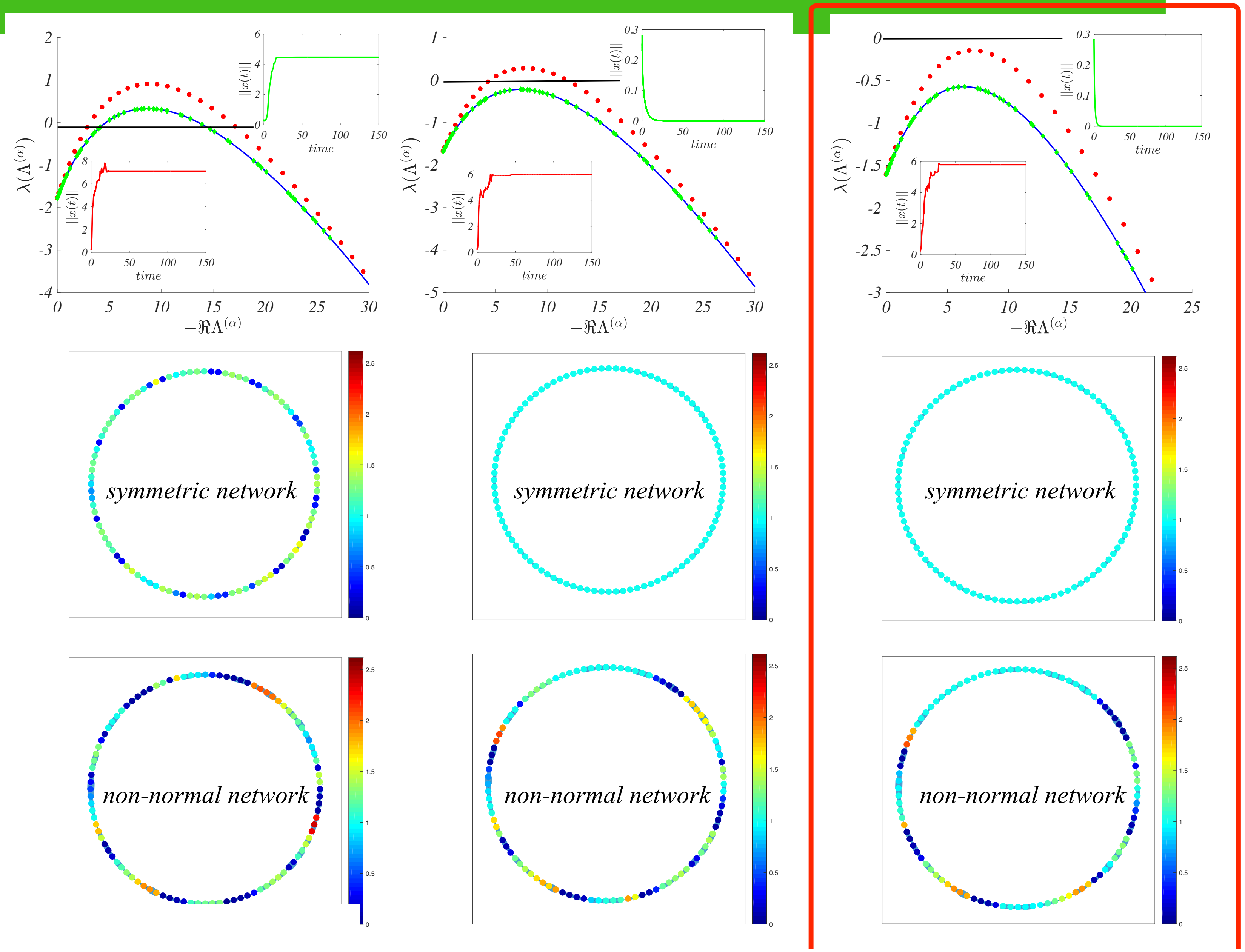


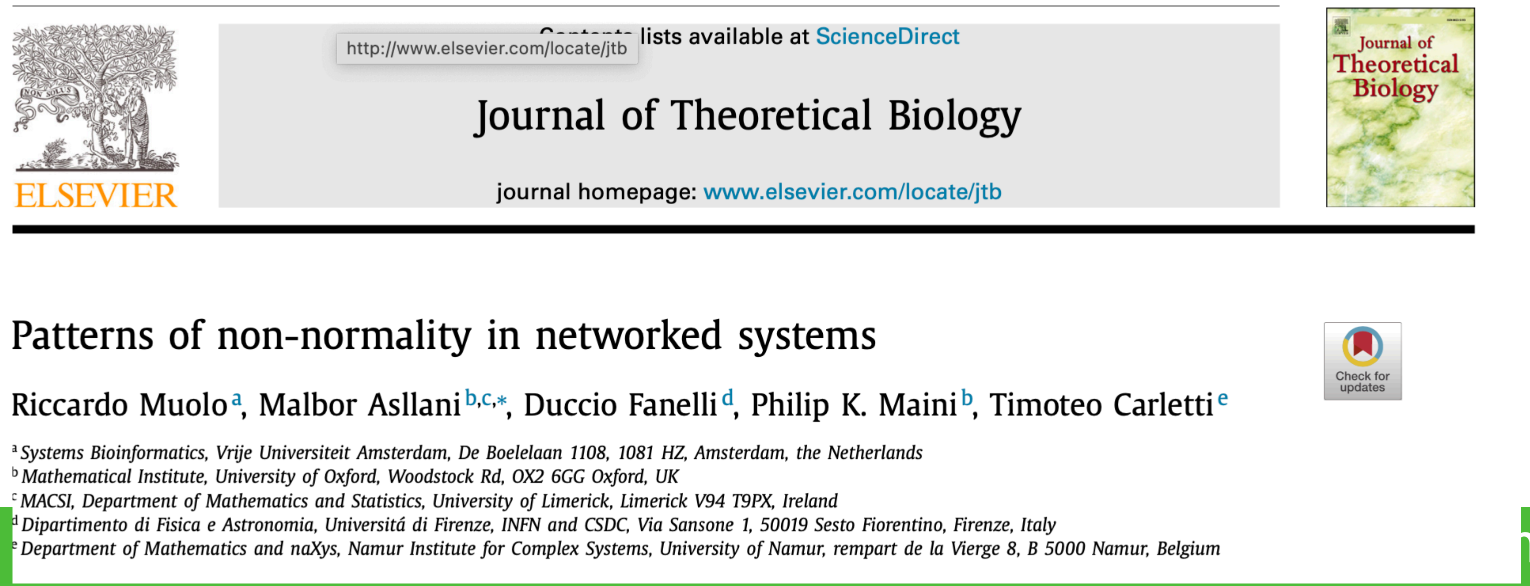
**Figure 3 | Instabilities on WS networks.** (**a**) Spectral plot of three Laplacians generated from the WS method for $p = 0.1$, $p = 0.2$ and $p = 0.8$ (blue triangles, red circles and green diamonds, respectively). In all cases, the network size is $\Omega = 100$. The coloured area indicates the instability region for the Brusselator model. (**b**) The real part of the dispersion relation for three choices of WS networks for $p = 0.1$, $p = 0.2$ and $p = 0.8$ (blue triangles, red circles and green diamonds, respectively), and network size $\Omega = 100$. The reaction parameters are $b = 9$, $c = 30$, $D_{\phi} = 1$ and $D_{\psi} = 7$.

Asllani M et al. 2014 The theory of pattern formation on directed networks. Nature Communication 5, 4517

# Dynamical Systems on non-normal Network



Journal of Theoretical Biology 480 (2019) 81–91


http://www.elsevier.com/locate/jtb
lists available at ScienceDirect

ELSEVIER

Journal of Theoretical Biology

journal homepage: www.elsevier.com/locate/jtb

Journal of Theoretical Biology

## Patterns of non-normality in networked systems

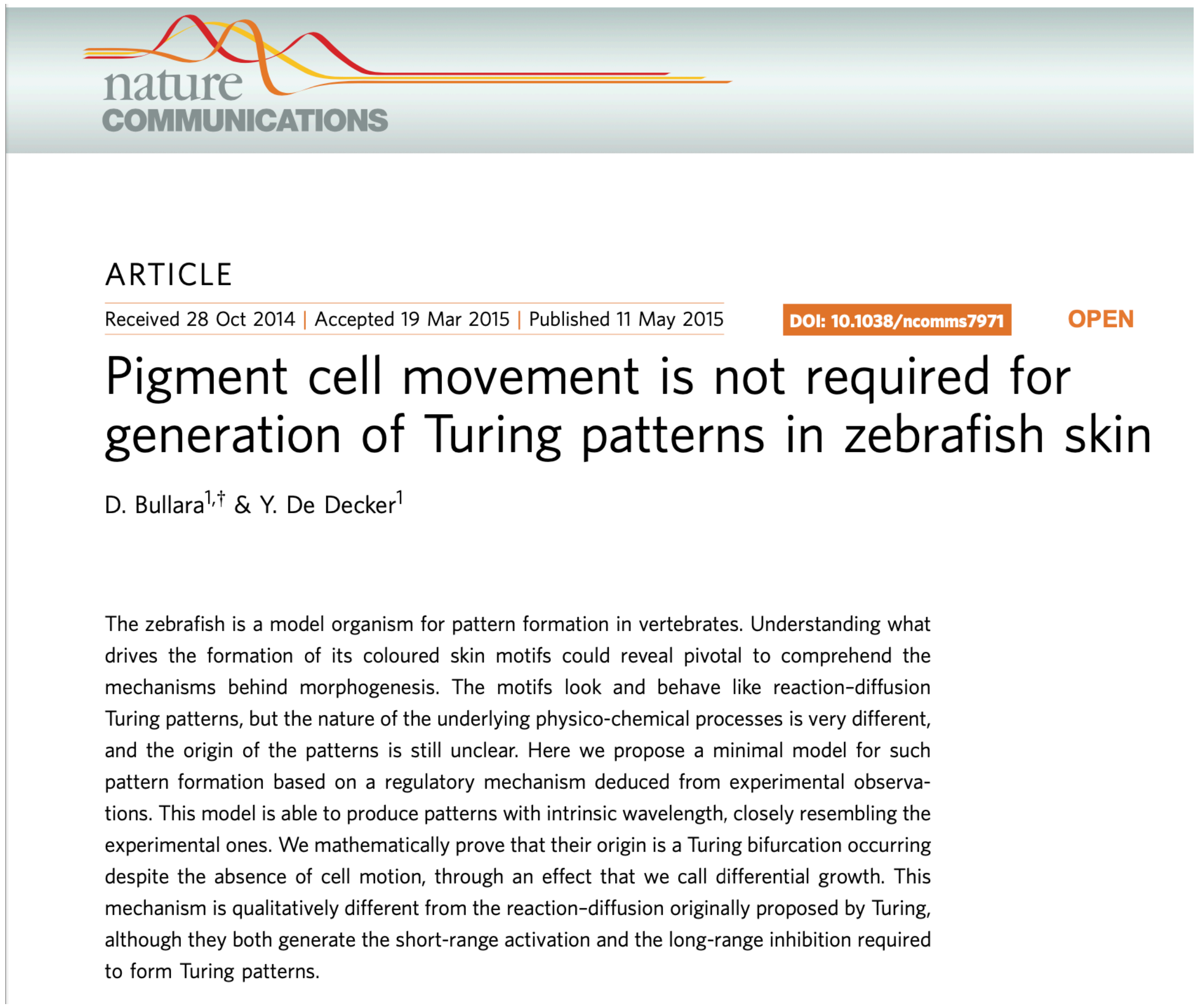
Check for updates

Riccardo Muolo[a], Malbor Asllani[b,c,*], Duccio Fanelli[d], Philip K. Maini[b], Timoteo Carletti[e]

[a] Systems Bioinformatics, Vrije Universiteit Amsterdam, De Boelelaan 1108, 1081 HZ, Amsterdam, the Netherlands
[b] Mathematical Institute, University of Oxford, Woodstock Rd, OX2 6GG Oxford, UK
[c] MACSI, Department of Mathematics and Statistics, University of Limerick, Limerick V94 T9PX, Ireland
[d] Dipartimento di Fisica e Astronomia, Universitá di Firenze, INFN and CSDC, Via Sansone 1, 50019 Sesto Fiorentino, Firenze, Italy
[e] Department of Mathematics and naXys, Namur Institute for Complex Systems, University of Namur, rempart de la Vierge 8, B 5000 Namur, Belgium

# Dynamical Systems on Network

# Adjacency coupling



nature COMMUNICATIONS

ARTICLE

Received 28 Oct 2014 | Accepted 19 Mar 2015 | Published 11 May 2015 DOI: 10.1038/ncomms7971 OPEN

## Pigment cell movement is not required for generation of Turing patterns in zebrafish skin

D. Bullara[1,†] & Y. De Decker[1]

The zebrafish is a model organism for pattern formation in vertebrates. Understanding what drives the formation of its coloured skin motifs could reveal pivotal to comprehend the mechanisms behind morphogenesis. The motifs look and behave like reaction–diffusion Turing patterns, but the nature of the underlying physico-chemical processes is very different, and the origin of the patterns is still unclear. Here we propose a minimal model for such pattern formation based on a regulatory mechanism deduced from experimental observations. This model is able to produce patterns with intrinsic wavelength, closely resembling the experimental ones. We mathematically prove that their origin is a Turing bifurcation occurring despite the absence of cell motion, through an effect that we call differential growth. This mechanism is qualitatively different from the reaction–diffusion originally proposed by Turing, although they both generate the short-range activation and the long-range inhibition required to form Turing patterns.

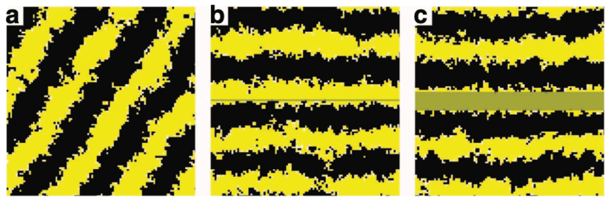


**Figure 1 | Stationary striped patterns observed in MC simulations.** The simulations are performed with a 100 × 100 square lattice with periodic boundary conditions and $b_M = d_X = d_M = 0$, $b_X = s_X = s_M = 1$, $l_X = 2.5$ and $h = 16$. They ran for $1 \times 10^9$ Monte Carlo steps, starting from an uniform initial condition without xantophores and melanophores. Yellow, black and white boxes represent X, M and S. (**a**) The pattern formation evolves freely. (**b**,**c**) We simulated the presence of an initial horizontal band of iridophores, which inhibit the growth of melanophores on top of them. The iridophores appear as a shaded band, the size of which is 1 cell (**b**) and 10 cells (**c**) wide.

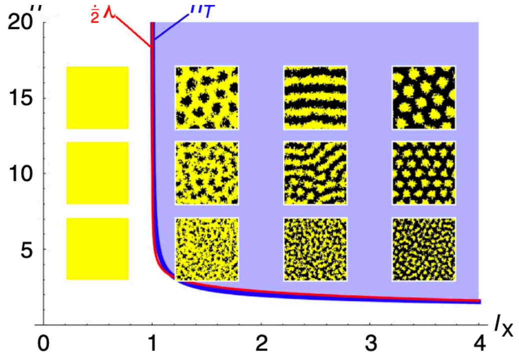
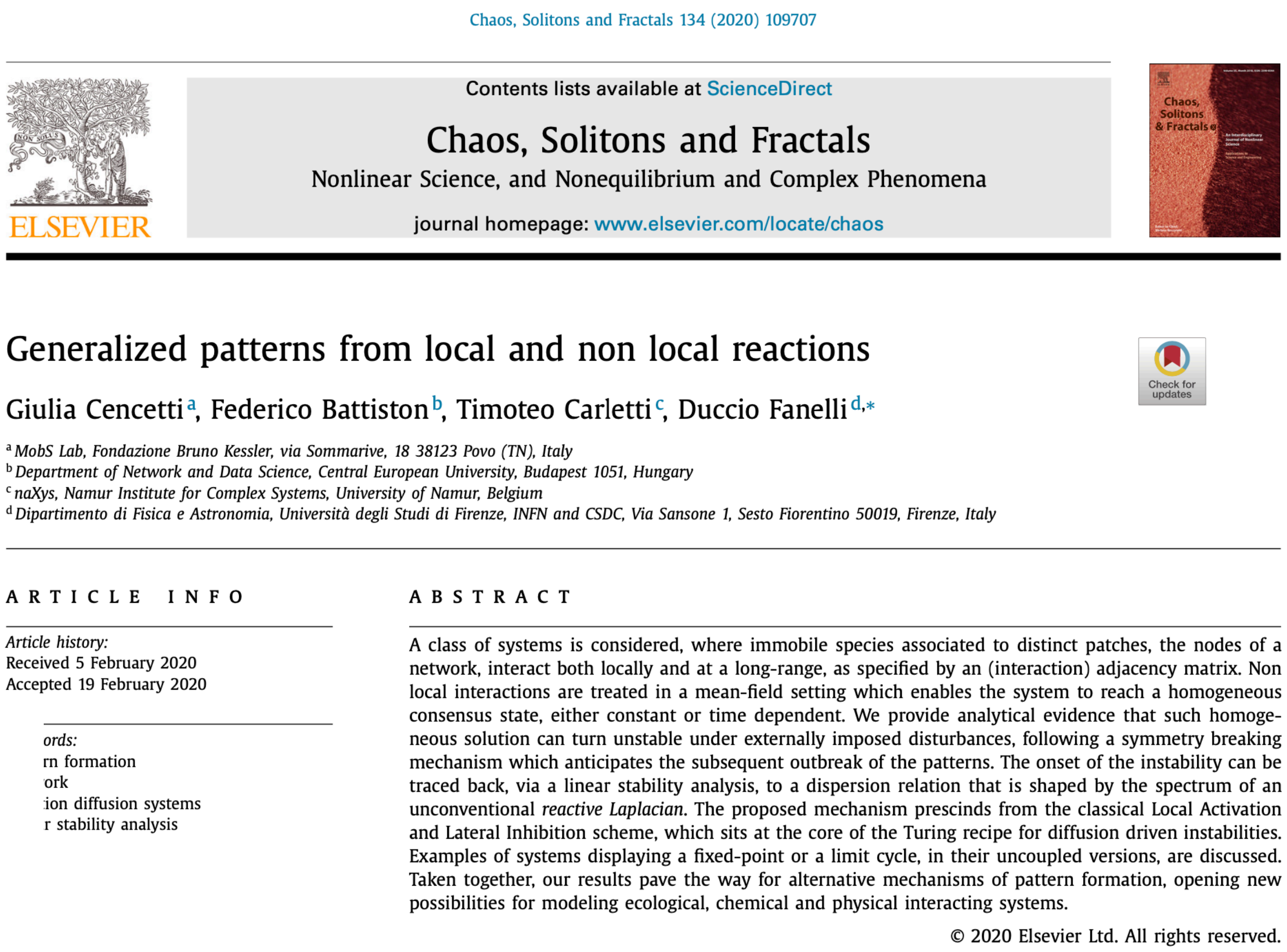


**Figure 2 | Comparison between the mean-field analytical bifurcation diagram and the MC simulations.** The values of the parameters are $b_M = d_X = d_M = 0$, $b_X = s_X = s_M = 1$, $h = \{5,10,15\}$ and $l_X = \{0.5,1.5,2.5,3.5\}$. The blue curve marks the critical values $h_T$, while the red curve is half of the critical wavelength $\lambda$.



Chaos, Solitons and Fractals 134 (2020) 109707

Contents lists available at ScienceDirect

Chaos, Solitons and Fractals

Nonlinear Science, and Nonequilibrium and Complex Phenomena

journal homepage: www.elsevier.com/locate/chaos

ELSEVIER

## Generalized patterns from local and non local reactions

Giulia Cencetti[a], Federico Battiston[b], Timoteo Carletti[c], Duccio Fanelli[d,*]

[a] MobS Lab, Fondazione Bruno Kessler, via Sommarive, 18 38123 Povo (TN), Italy
[b] Department of Network and Data Science, Central European University, Budapest 1051, Hungary
[c] naXys, Namur Institute for Complex Systems, University of Namur, Belgium
[d] Dipartimento di Fisica e Astronomia, Università degli Studi di Firenze, INFN and CSDC, Via Sansone 1, Sesto Fiorentino 50019, Firenze, Italy

ARTICLE INFO

*Article history:*
Received 5 February 2020
Accepted 19 February 2020

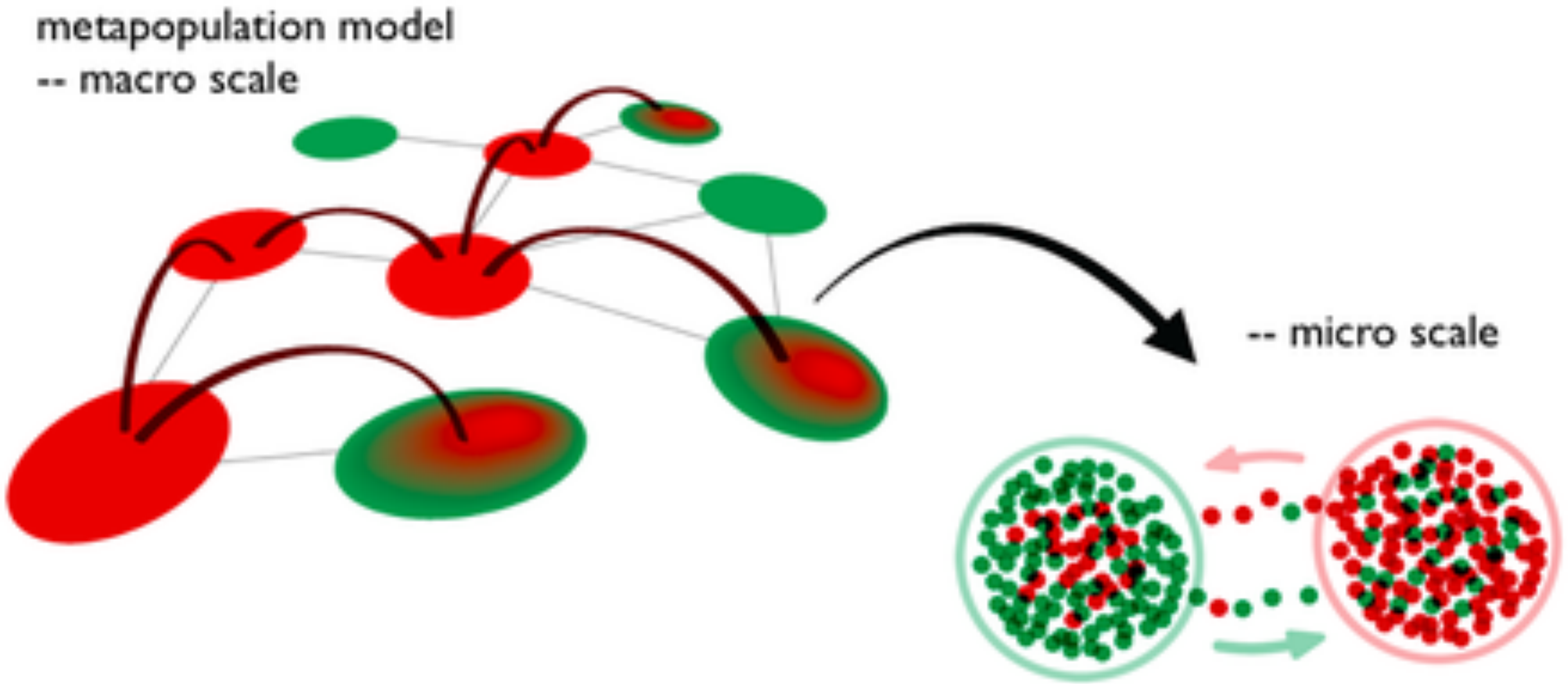
*ords:*
rn formation
ork
ion diffusion systems
r stability analysis

ABSTRACT

A class of systems is considered, where immobile species associated to distinct patches, the nodes of a network, interact both locally and at a long-range, as specified by an (interaction) adjacency matrix. Non local interactions are treated in a mean-field setting which enables the system to reach a homogeneous consensus state, either constant or time dependent. We provide analytical evidence that such homogeneous solution can turn unstable under externally imposed disturbances, following a symmetry breaking mechanism which anticipates the subsequent outbreak of the patterns. The onset of the instability can be traced back, via a linear stability analysis, to a dispersion relation that is shaped by the spectrum of an unconventional *reactive Laplacian*. The proposed mechanism prescinds from the classical Local Activation and Lateral Inhibition scheme, which sits at the core of the Turing recipe for diffusion driven instabilities. Examples of systems displaying a fixed-point or a limit cycle, in their uncoupled versions, are discussed. Taken together, our results pave the way for alternative mechanisms of pattern formation, opening new possibilities for modeling ecological, chemical and physical interacting systems.

© 2020 Elsevier Ltd. All rights reserved.

# Dynamical Systems on Network



## Metapopulation models

e.g. in the framework of ecology:
*May R., Will a large complex system be stable?*
Nature, 238, pp. 413, (1972)

Interactions occur at each node and among nearby ones

# Dynamical Systems on Network

Assume each isolated system converges to the same stable periodic solution

$$\lim_{t\to\infty} \vec{x}_i(t) = \vec{s}(t) \quad \forall i$$

Assume a adjacency coupling

$$\dot{\vec{x}}_i \sim \frac{1}{k_i} \sum_j A_{ij} F(\vec{x}_i, \vec{x}_j)$$

# Dynamical Systems on Network

Assume each isolated system converges to the same stable periodic solution

$$\lim_{t\to\infty} \vec{x}_i(t) = \vec{s}(t) \quad \forall i$$

Assume a adjacency coupling

$$\dot{\vec{x}}_i \sim \frac{1}{k_i}\sum_j A_{ij} F(\vec{x}_i, \vec{x}_j)$$

$$= \frac{1}{k_i}\sum_j A_{ij} F(\vec{x}_i, \vec{x}_j) - F(\vec{x}_i, \vec{x}_i) + F(\vec{x}_i, \vec{x}_i) =$$

$$= \sum_j \left(\frac{A_{ij}}{k_i} - \delta_{ij}\right) F(\vec{x}_i, \vec{x}_j) + F(\vec{x}_i, \vec{x}_i) = \sum_j \mathcal{L}_{ij} F(\vec{x}_i, \vec{x}_j) + F(\vec{x}_i, \vec{x}_i)$$

# Dynamical Systems on Network

$\mathcal{L}_{ij} = \frac{A_{ij}}{k_i} - \delta_{ij}$ is the (consensus or reactive) Laplace matrix

$\mathcal{L} = \mathbf{D}^{-1}\mathbf{A} - \mathbf{I}$ It is not symmetric, however

# Dynamical Systems on Network

$$\mathcal{L}_{ij} = \frac{A_{ij}}{k_i} - \delta_{ij}$$ is the (consensus or reactive) Laplace matrix

$$\mathcal{L} = \mathbf{D}^{-1}\mathbf{A} - \mathbf{I}$$ It is not symmetric, however

$$\mathbf{L}^{sym} := \mathbf{D}^{-1/2}\left(\mathbf{A} - \mathbf{D}\right)\mathbf{D}^{-1/2}$$

is symmetric and non-positive definite

# Dynamical Systems on Network

$$\mathcal{L}_{ij} = \frac{A_{ij}}{k_i} - \delta_{ij}$$ is the (consensus or reactive) Laplace matrix

$$\mathcal{L} = \mathbf{D}^{-1}\mathbf{A} - \mathbf{I}$$ It is not symmetric, however

$$\mathbf{L}^{sym} := \mathbf{D}^{-1/2}\left(\mathbf{A} - \mathbf{D}\right)\mathbf{D}^{-1/2}$$

is symmetric and non-positive definite

$$\mathcal{L} = \mathbf{D}^{-1/2}\mathbf{L}^{sym}\mathbf{D}^{1/2}$$

hence they have the same eigenvalues

$$-2 < \Lambda^{(\alpha)} \leq 0$$

$$\sum_j \mathcal{L}_{ij} = 0 \qquad \Lambda^{(1)} = 0 \qquad \vec{\phi}^{(1)} = (1, \dots, 1)^\top / \sqrt{n}$$

# A second relevant example : the Stuart-Landau

$$\frac{dz}{dt} = \sigma z - \beta |z|^2 z \qquad \sigma = \sigma_{\Re} + i\sigma_{\Im} \quad \beta = \beta_{\Re} + i\beta_{\Im}$$

Periodic solution $$\hat{z}(t) = \sqrt{\frac{\sigma_{\Re}}{\beta_{\Re}}} e^{i\omega t} \qquad \omega = \sigma_{\Im} - \beta_{\Im}\frac{\sigma_{\Re}}{\beta_{\Re}}$$

# Dynamical Systems on Network

## A second relevant example : the Stuart-Landau

$$\frac{dz}{dt} = \sigma z - \beta |z|^2 z \qquad \sigma = \sigma_{\Re} + i\sigma_{\Im} \quad \beta = \beta_{\Re} + i\beta_{\Im}$$

**Periodic solution**

$$\hat{z}(t) = \sqrt{\frac{\sigma_{\Re}}{\beta_{\Re}}} e^{i\omega t} \qquad \omega = \sigma_{\Im} - \beta_{\Im} \frac{\sigma_{\Re}}{\beta_{\Re}}$$

**Stability**

$$z(t) = \hat{z}(t) \left(1 + \rho(t)\right) e^{i\theta(t)}$$

$$|\rho(t)| \ll 1$$

$$|\theta(t)| \ll 1$$

$$\frac{d}{dt}\begin{pmatrix} \rho \\ \theta \end{pmatrix} = \begin{pmatrix} -2\sigma_{\Re} & 0 \\ -2\beta_{\Im}\frac{\sigma_{\Re}}{\beta_{\Re}} & 0 \end{pmatrix} \begin{pmatrix} \rho \\ \theta \end{pmatrix}$$

$$\boxed{\sigma_{\Re} > 0}$$

$$\boxed{\beta_{\Re} > 0}$$

# Dynamical Systems on Network

## A second relevant example : the Stuart-Landau

$$\frac{dz_j}{dt} = \sigma z_j - \beta |z_j|^2 z_j \qquad \sigma = \sigma_{\Re} + i\sigma_{\Im} \qquad \beta = \beta_{\Re} + i\beta_{\Im}$$

Coupling

$$\frac{dz_j}{dt} = \frac{\sigma}{k_j} \sum_{\ell} A_{j\ell} z_\ell - \beta z_j |z_j|^2$$

$$\frac{dz_j}{dt} = \sigma \sum_{\ell} \mathcal{L}_{j\ell} z_\ell + \sigma z_j - \beta z_j |z_j|^2$$

# Dynamical Systems on Network

## A second relevant example : the Stuart-Landau

$$\frac{dz_j}{dt} = \sigma z_j - \beta |z_j|^2 z_j \qquad \sigma = \sigma_{\Re} + i\sigma_{\Im} \qquad \beta = \beta_{\Re} + i\beta_{\Im}$$

Coupling

$$\frac{dz_j}{dt} = \frac{\sigma}{k_j} \sum_{\ell} A_{j\ell} z_\ell - \beta z_j |z_j|^2$$

$$\frac{dz_j}{dt} = \sigma \sum_{\ell} \mathcal{L}_{j\ell} z_\ell + \sigma z_j - \beta z_j |z_j|^2$$

$$\hat{z}(t) = \sqrt{\frac{\sigma_{\Re}}{\beta_{\Re}}} e^{i\omega t}$$

is also a solution of the coupled system

# Dynamical Systems on Network

Stability $$z_j(t) = \hat{z}(t)\left(1+\rho_j(t)\right)e^{i\theta_j(t)}$$

$$|\rho_j(t)| \ll 1 \qquad |\theta_j(t)| \ll 1$$

$$\frac{d}{dt}\begin{pmatrix}\rho_j\\ \theta_j\end{pmatrix} = \begin{pmatrix}-2\sigma_{\Re} & 0\\ -2\beta_{\Im}\frac{\sigma_{\Re}}{\beta_{\Re}} & 0\end{pmatrix}\begin{pmatrix}\rho_j\\ \theta_j\end{pmatrix} + \sum_{\ell}\mathcal{L}_{j\ell}\begin{pmatrix}\sigma_{\Re} & -\sigma_{\Im}\\ \sigma_{\Im} & \sigma_{\Re}\end{pmatrix}\begin{pmatrix}\rho_\ell\\ \theta_\ell\end{pmatrix}$$

# Dynamical Systems on Network

Stability $z_j(t) = \hat{z}(t)\,(1+\rho_j(t))\,e^{i\theta_j(t)}$

$|\rho_j(t)| \ll 1 \qquad |\theta_j(t)| \ll 1$

$$\frac{d}{dt}\begin{pmatrix}\rho_j\\ \theta_j\end{pmatrix} = \begin{pmatrix}-2\sigma_{\Re} & 0\\ -2\beta_{\Im}\frac{\sigma_{\Re}}{\beta_{\Re}} & 0\end{pmatrix}\begin{pmatrix}\rho_j\\ \theta_j\end{pmatrix} + \sum_{\ell}\mathcal{L}_{j\ell}\begin{pmatrix}\sigma_{\Re} & -\sigma_{\Im}\\ \sigma_{\Im} & \sigma_{\Re}\end{pmatrix}\begin{pmatrix}\rho_\ell\\ \theta_\ell\end{pmatrix}$$

Use the Laplace eigenbasis $\rho_j = \sum_\alpha \rho^\alpha \phi_j^{(\alpha)} \qquad \theta_j = \sum_\alpha \theta^\alpha \phi_j^{(\alpha)}$

$$\frac{d}{dt}\begin{pmatrix}\rho_\alpha\\ \theta_\alpha\end{pmatrix} = \underbrace{\left[\begin{pmatrix}-2\sigma_{\Re} & 0\\ -2\beta_{\Im}\frac{\sigma_{\Re}}{\beta_{\Re}} & 0\end{pmatrix} + \Lambda^{(\alpha)}\begin{pmatrix}\sigma_{\Re} & -\sigma_{\Im}\\ \sigma_{\Im} & \sigma_{\Re}\end{pmatrix}\right]}_{\mathbf{J}_\alpha = \mathbf{J}_0 + \Lambda^{(\alpha)}\mathbf{J}_2}\begin{pmatrix}\rho_\alpha\\ \theta_\alpha\end{pmatrix}$$

# Dynamical Systems on Network

$$\mathbf{J}_\alpha = \mathbf{J}_0 + \Lambda^{(\alpha)}\mathbf{J}_2 \qquad \lambda_\alpha^2 - \lambda_\alpha \mathrm{tr}(\mathbf{J}_\alpha) + \det(\mathbf{J}_\alpha) = 0$$

instability $\mathrm{tr}(\mathbf{J}_\alpha) \geq 0 \text{ or } \det(\mathbf{J}_\alpha) \leq 0$ for some $\alpha$

timoteo.carletti@unamur.be

# Dynamical Systems on Network

$$\mathbf{J}_\alpha = \mathbf{J}_0 + \Lambda^{(\alpha)}\mathbf{J}_2 \qquad \lambda_\alpha^2 - \lambda_\alpha \mathrm{tr}(\mathbf{J}_\alpha) + \det(\mathbf{J}_\alpha) = 0$$

instability $\mathrm{tr}(\mathbf{J}_\alpha) \geq 0 \text{ or } \det(\mathbf{J}_\alpha) \leq 0$ for some $\alpha$

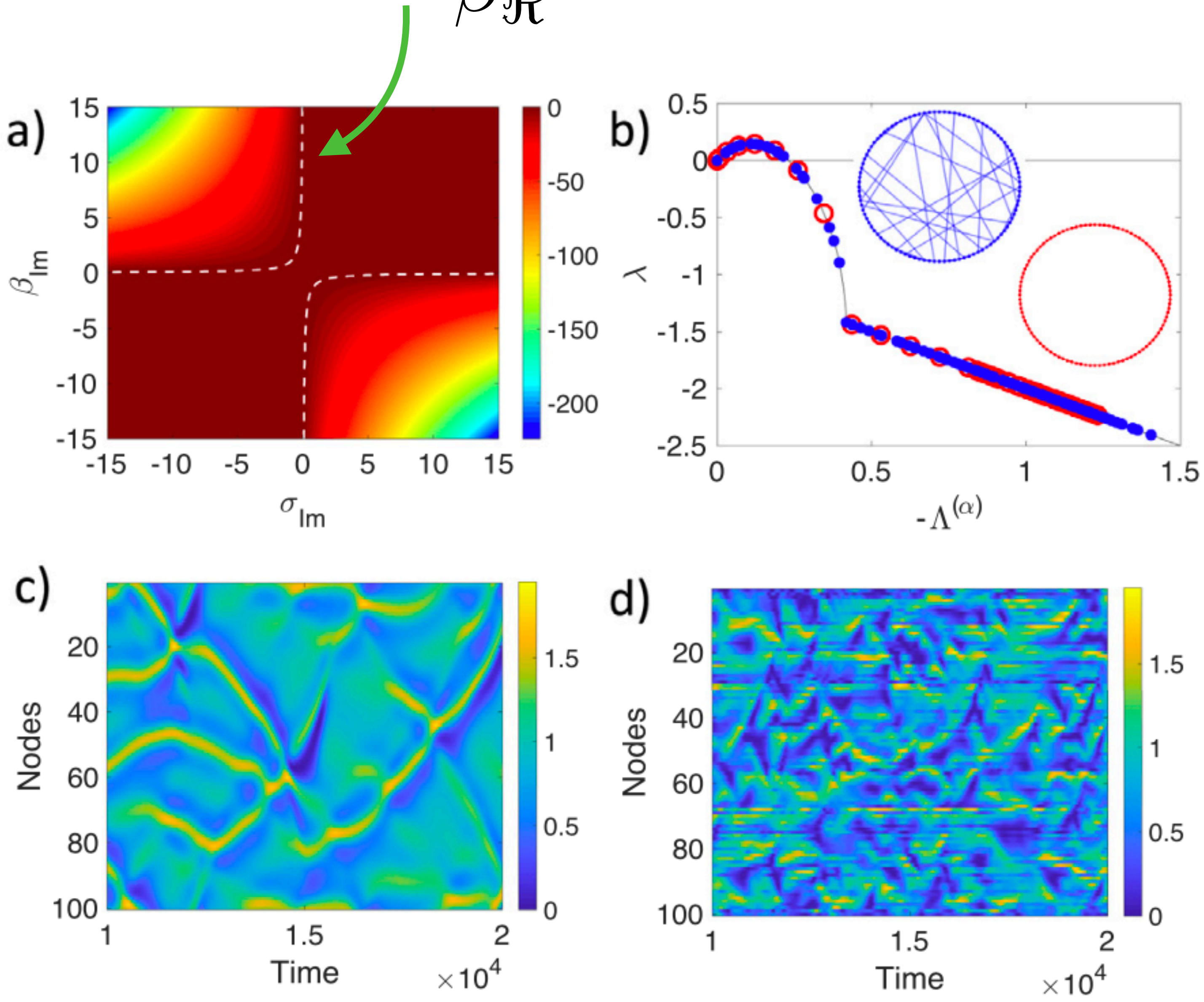
but $\mathrm{tr}(\mathbf{J}_\alpha) = \mathrm{tr}(\mathbf{J}_0) + \Lambda^{(\alpha)} 2\sigma_\Re < \mathrm{tr}(\mathbf{J}_0) < 0$

$$\Lambda^{(\alpha)} > \frac{2\sigma_\Re}{\sigma_\Re^2 + \sigma_\Im^2}\left(\sigma_\Re + \frac{\sigma_\Im \beta_\Im}{\beta_\Re}\right)$$

$$\sigma_\Im \beta_\Im < -\sigma_\Re \beta_\Re$$

# Dynamical Systems on Network

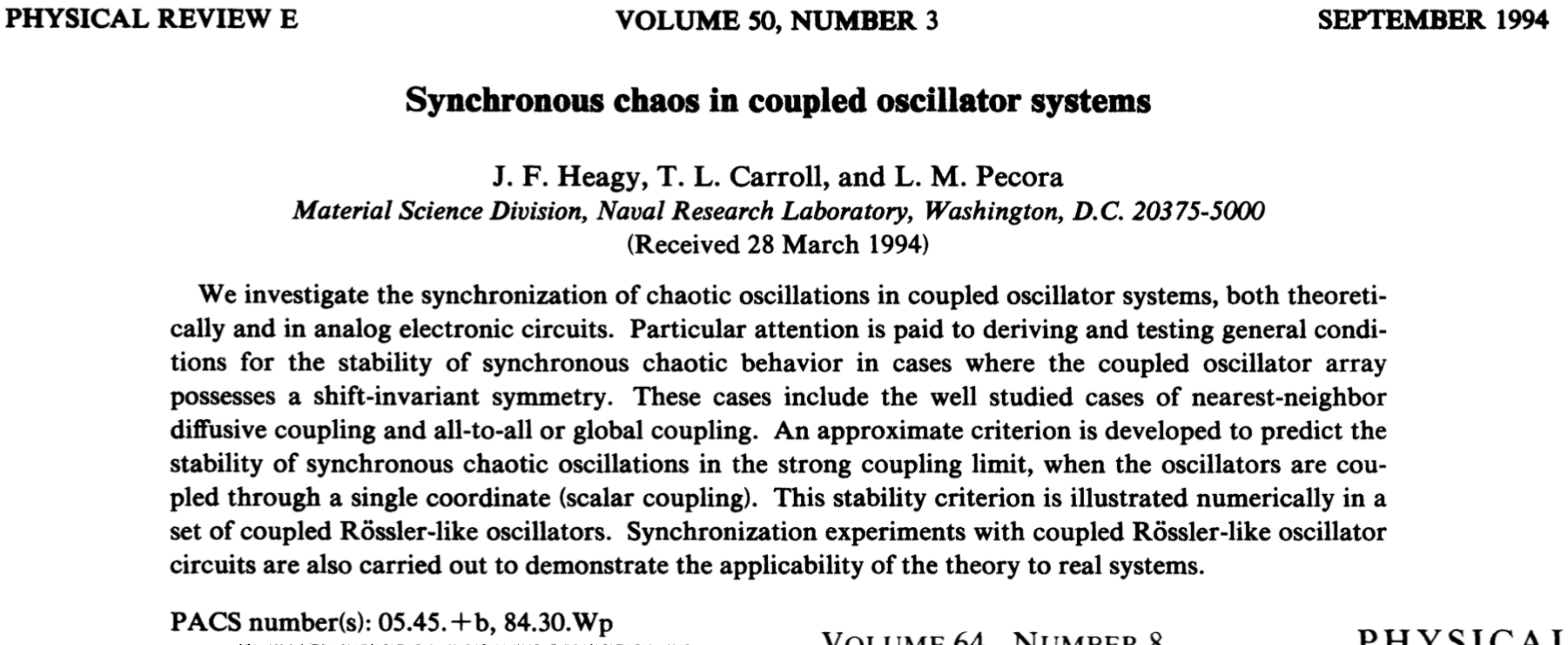
$$\sigma_{\Re} = \beta_{\Re} = 1 \qquad \sigma_{\Re} + \frac{\sigma_{\Im}\beta_{\Im}}{\beta_{\Re}} = 0$$



Lattice (k=6) Watts-Strogatz (k=6)

Cencetti G et al. 2020 Generalized patterns from local and non local reactions, Chaos 134, 109707

# Dynamical Systems on Network



PHYSICAL REVIEW E VOLUME 50, NUMBER 3 SEPTEMBER 1994

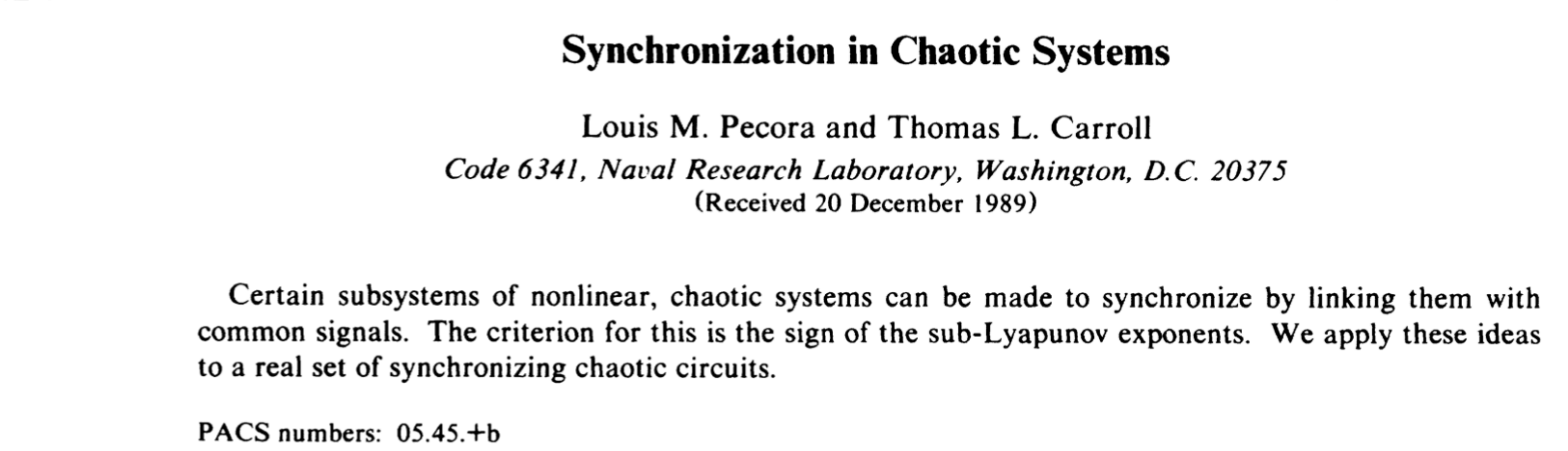
### Synchronous chaos in coupled oscillator systems

J. F. Heagy, T. L. Carroll, and L. M. Pecora
*Material Science Division, Naval Research Laboratory, Washington, D.C. 20375-5000*
(Received 28 March 1994)

We investigate the synchronization of chaotic oscillations in coupled oscillator systems, both theoretically and in analog electronic circuits. Particular attention is paid to deriving and testing general conditions for the stability of synchronous chaotic behavior in cases where the coupled oscillator array possesses a shift-invariant symmetry. These cases include the well studied cases of nearest-neighbor diffusive coupling and all-to-all or global coupling. An approximate criterion is developed to predict the stability of synchronous chaotic oscillations in the strong coupling limit, when the oscillators are coupled through a single coordinate (scalar coupling). This stability criterion is illustrated numerically in a set of coupled Rössler-like oscillators. Synchronization experiments with coupled Rössler-like oscillator circuits are also carried out to demonstrate the applicability of the theory to real systems.

PACS number(s): 05.45.+b, 84.30.Wp

**Chaotic behaviour**



VOLUME 64, NUMBER 8 PHYSICAL REVIEW LETTERS 19 FEBRUARY 1990

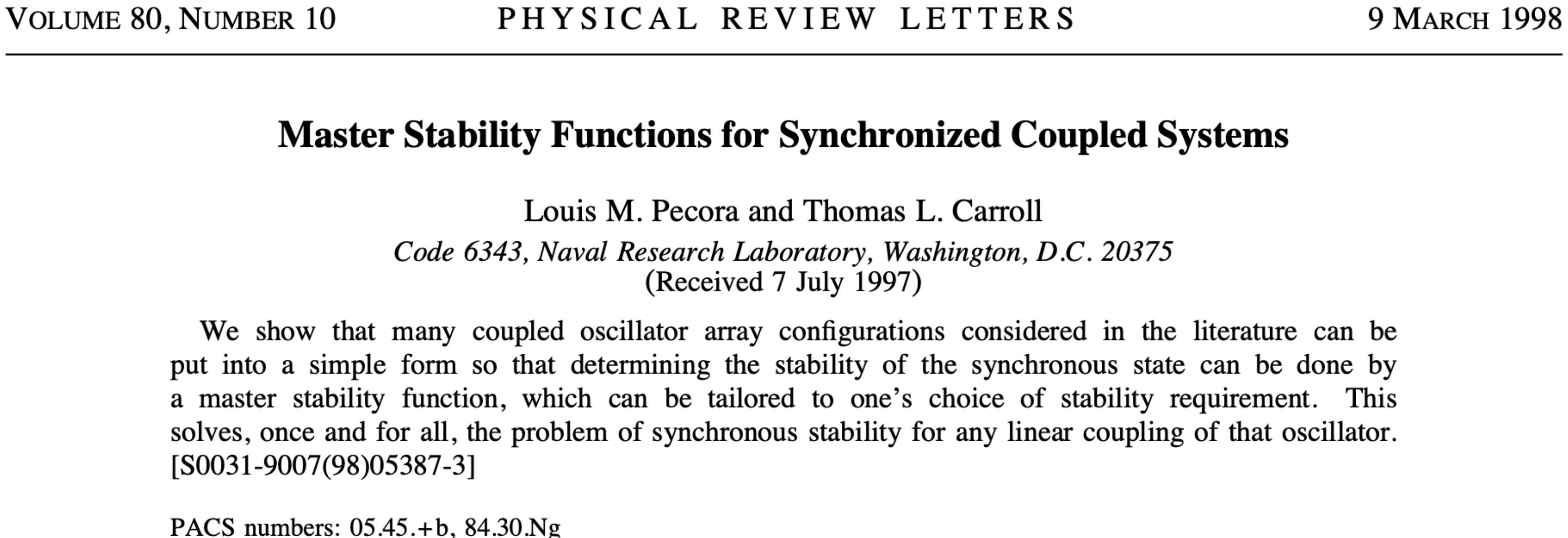
### Synchronization in Chaotic Systems

Louis M. Pecora and Thomas L. Carroll
*Code 6341, Naval Research Laboratory, Washington, D.C. 20375*
(Received 20 December 1989)

Certain subsystems of nonlinear, chaotic systems can be made to synchronize by linking them with common signals. The criterion for this is the sign of the sub-Lyapunov exponents. We apply these ideas to a real set of synchronizing chaotic circuits.

PACS numbers: 05.45.+b

**Master Stability Function**



VOLUME 80, NUMBER 10 PHYSICAL REVIEW LETTERS 9 MARCH 1998

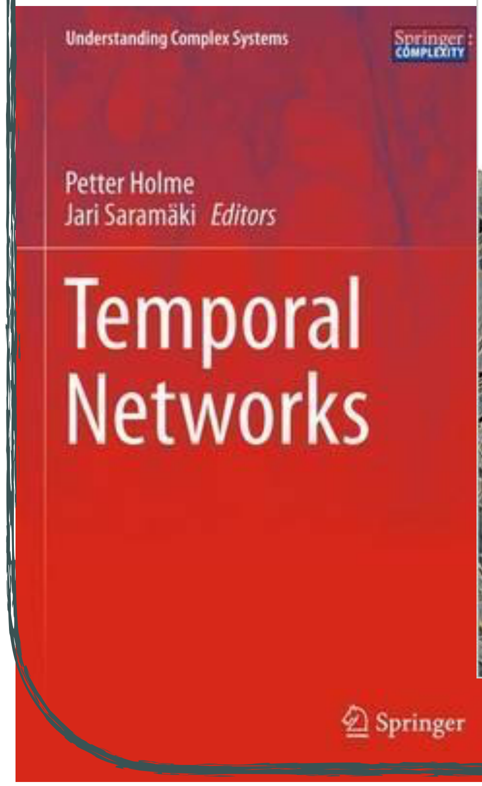
### Master Stability Functions for Synchronized Coupled Systems

Louis M. Pecora and Thomas L. Carroll
*Code 6343, Naval Research Laboratory, Washington, D.C. 20375*
(Received 7 July 1997)

We show that many coupled oscillator array configurations considered in the literature can be put into a simple form so that determining the stability of the synchronous state can be done by a master stability function, which can be tailored to one's choice of stability requirement. This solves, once and for all, the problem of synchronous stability for any linear coupling of that oscillator. [S0031-9007(98)05387-3]

PACS numbers: 05.45.+b, 84.30.Ng

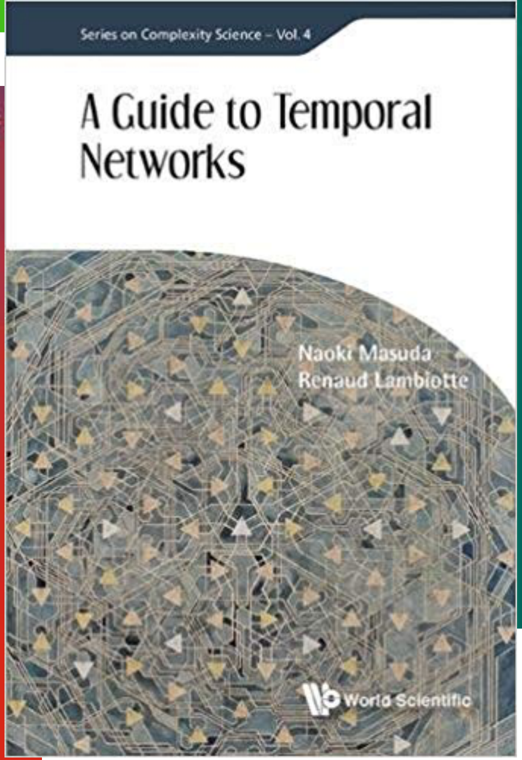
# Beyond networks


Understanding Complex Systems

Petter Holme
Jari Saramäki *Editors*

Temporal Networks

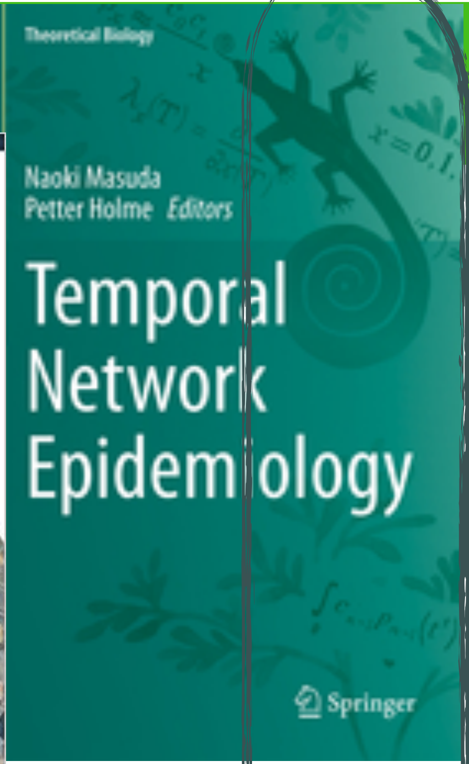
Springer


Series on Complexity Science – Vol. 4

A Guide to Temporal Networks

Naoki Masuda
Renaud Lambiotte

World Scientific


Theoretical Biology

Naoki Masuda
Petter Holme *Editors*

Temporal Network Epidemiology

Springer


**nature communications**

Explore content ⌄ About the journal ⌄ Publish with us ⌄

nature › nature communications › articles › article

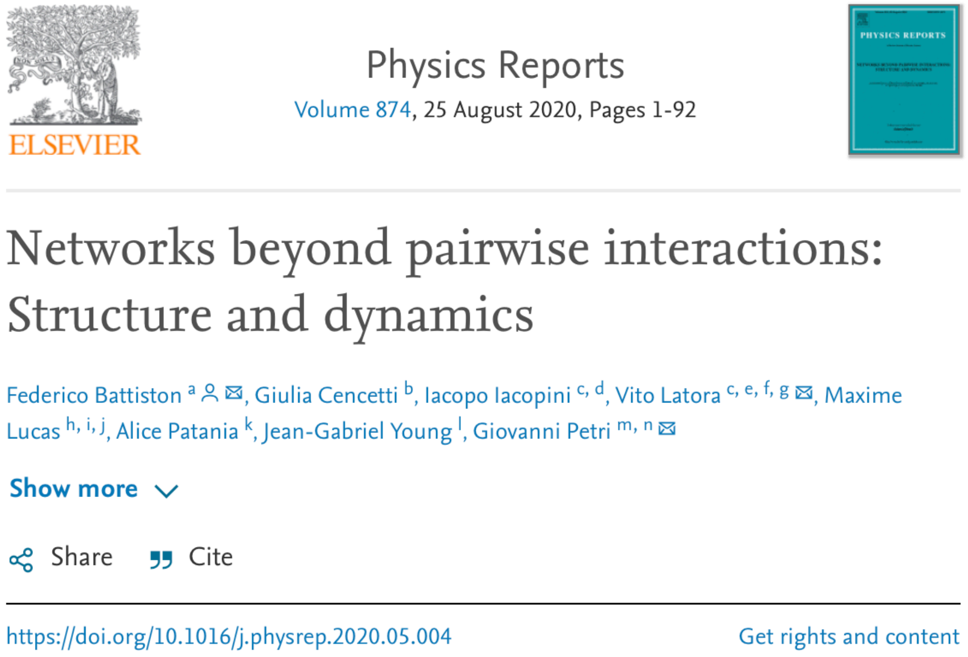
Article | Open Access | Published: 06 June 2019

**Simplicial models of social contagion**

Iacopo Iacopini, Giovanni Petri, Alain Barrat & Vito Latora

*Nature Communications* **10**, Article number: 2485 (2019) | Cite this article

**14k** Ac


ELSEVIER

Physics Reports

Volume 874, 25 August 2020, Pages 1-92

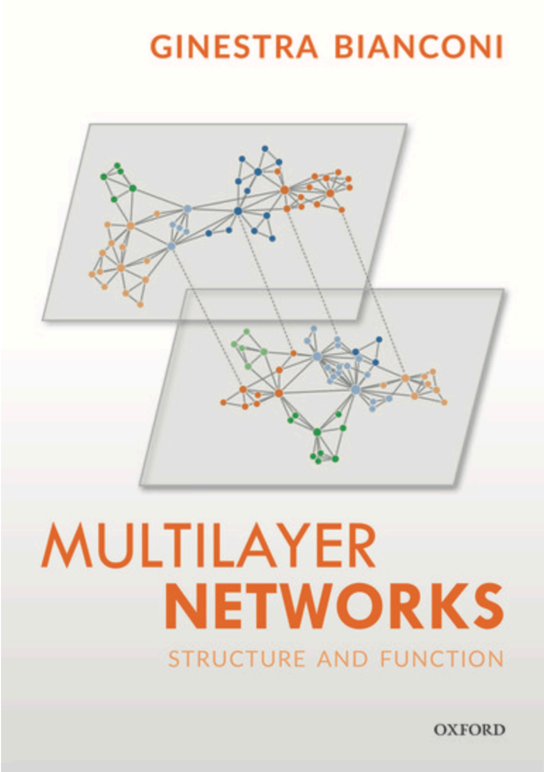
Networks beyond pairwise interactions: Structure and dynamics

Federico Battiston [a], Giulia Cencetti [b], Iacopo Iacopini [c, d], Vito Latora [c, e, f, g], Maxime Lucas [h, i, j], Alice Patania [k], Jean-Gabriel Young [l], Giovanni Petri [m, n]

Show more ⌄

Share Cite

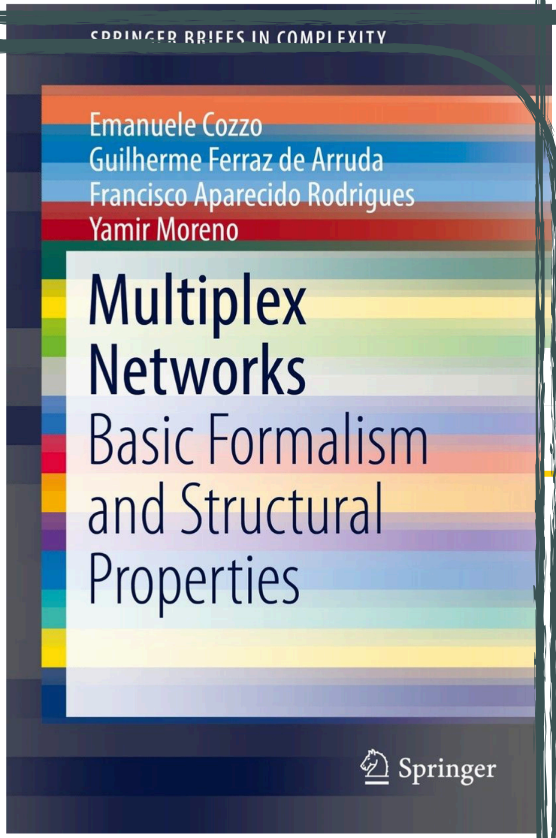
https://doi.org/10.1016/j.physrep.2020.05.004 Get rights and content


GINESTRA BIANCONI

MULTILAYER NETWORKS

STRUCTURE AND FUNCTION

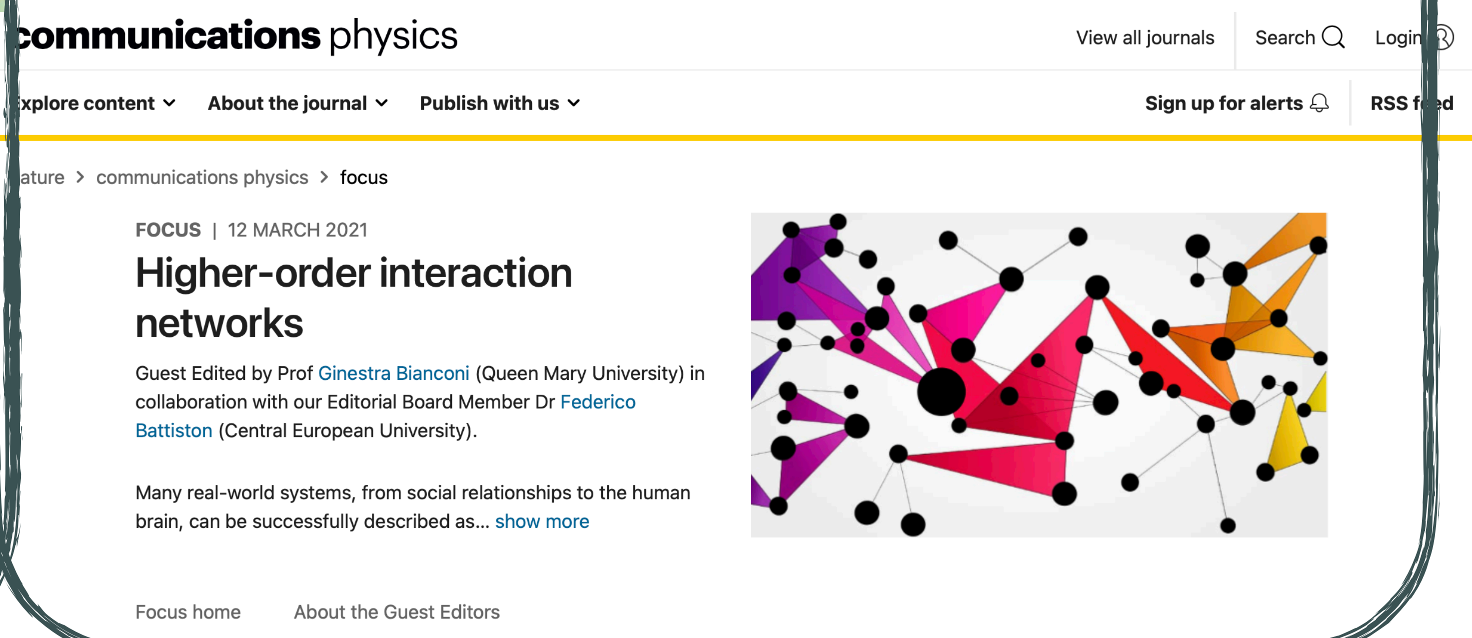
OXFORD


SPRINGER BRIEFS IN COMPLEXITY

Emanuele Cozzo
Guilherme Ferraz de Arruda
Francisco Aparecido Rodrigues
Yamir Moreno

Multiplex Networks
Basic Formalism and Structural Properties

Springer


**communications** physics

View all journals Search Login

xplore content ⌄ About the journal ⌄ Publish with us ⌄ Sign up for alerts RSS feed

ature › communications physics › focus

FOCUS | 12 MARCH 2021

Higher-order interaction networks

Guest Edited by Prof Ginestra Bianconi (Queen Mary University) in collaboration with our Editorial Board Member Dr Federico Battiston (Central European University).

Many real-world systems, from social relationships to the human brain, can be successfully described as... show more

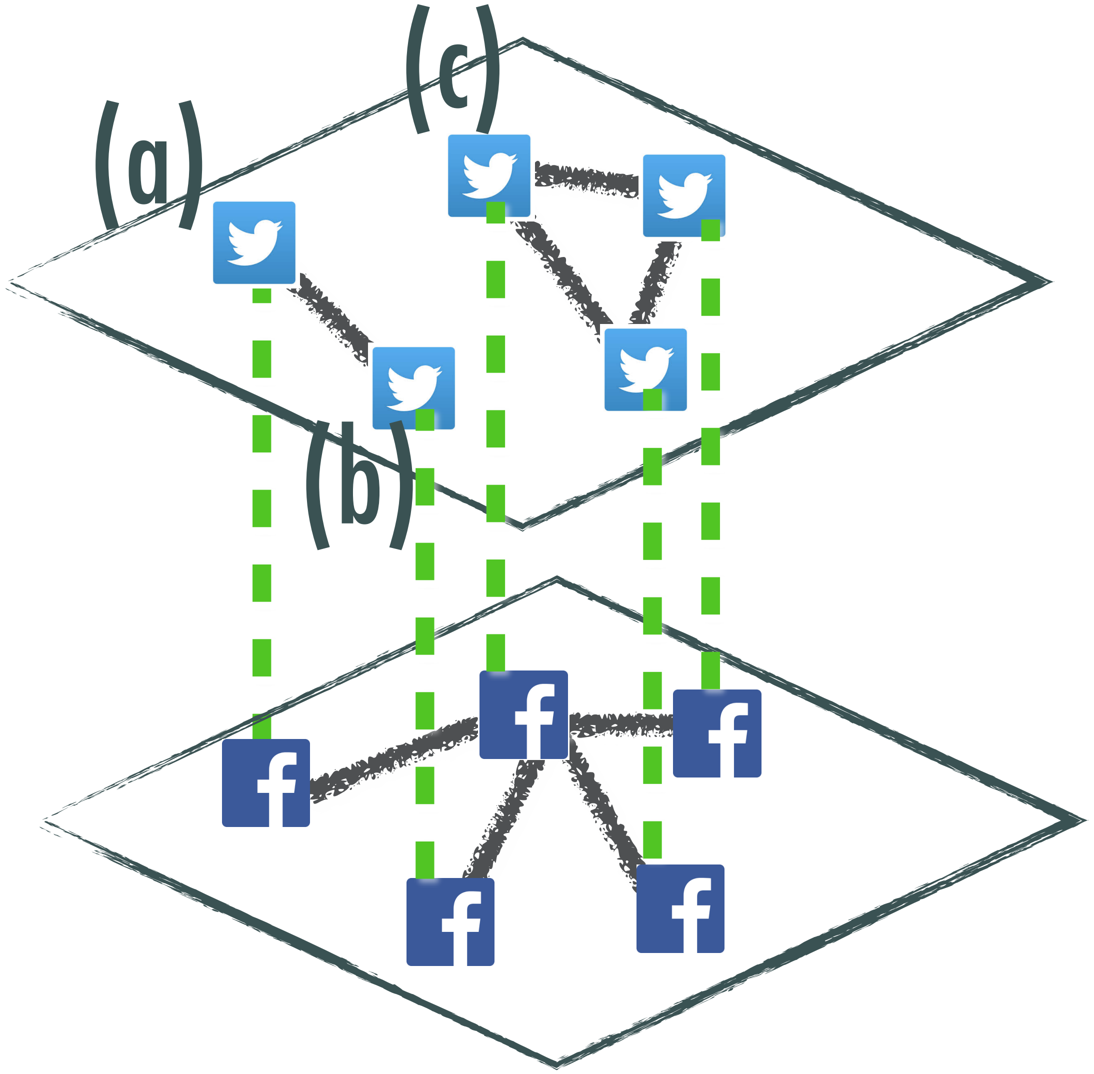
Focus home About the Guest Editors

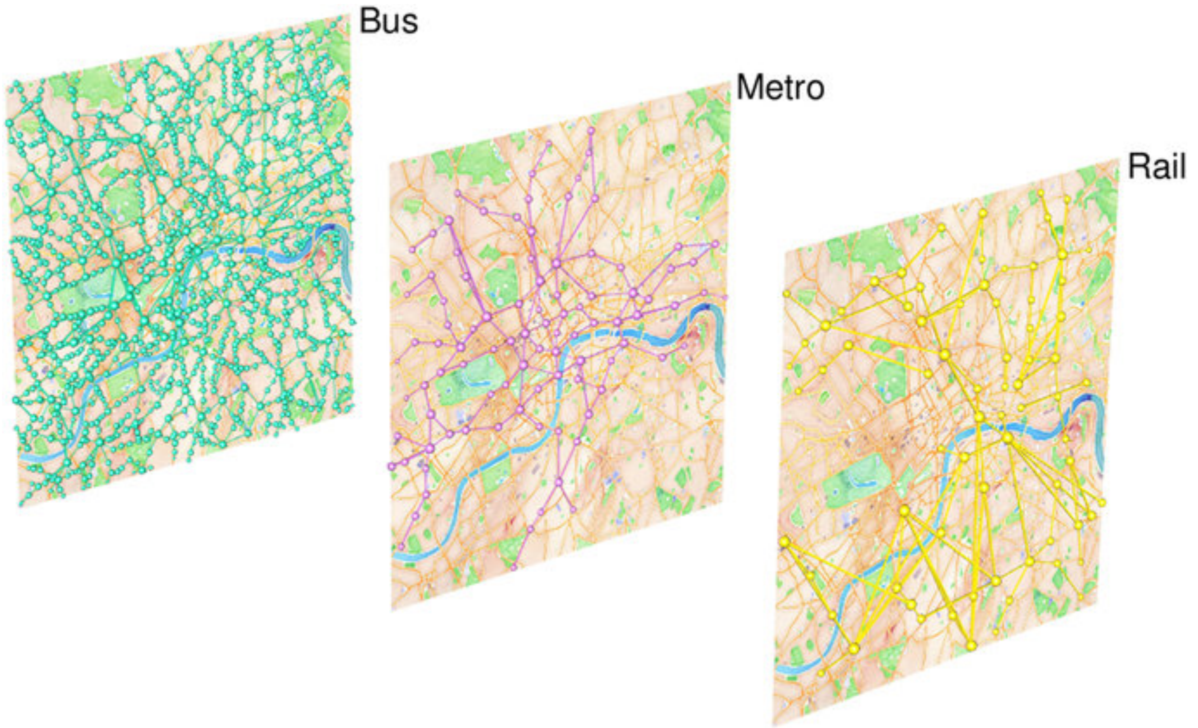
# Multiplexes / Multilayers

Social networks

layers=different social networks

nodes=same agent in each SN



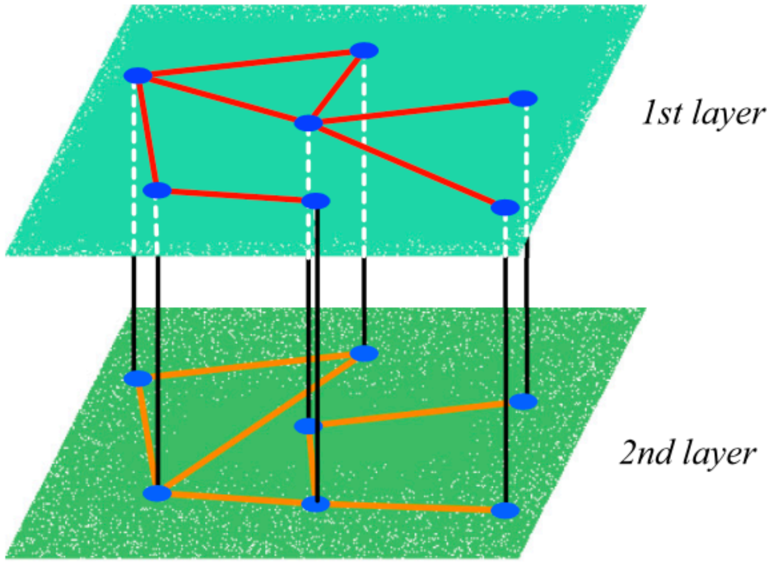


Transportation networks

layers=different modalities

nodes=same spatial location

# Multiplexes / Multilayers



$$\dot{u}_i^K = f\left(u_i^K, v_i^K\right) + D_u^K \sum_{j=1}^{\Omega} L_{ij}^K u_j^K + D_u^{12}\left(u_i^{K+1} - u_i^K\right),$$

$$\dot{v}_i^K = g\left(u_i^K, v_i^K\right) + D_v^K \sum_{j=1}^{\Omega} L_{ij}^K v_j^K + D_v^{12}\left(v_i^{K+1} - v_i^K\right)$$

$$\tilde{\mathcal{J}} = \begin{pmatrix} f_u \mathbf{I}_{2\Omega} + \mathcal{L}_u + D_u^{12}\mathcal{I} & f_v \mathbf{I}_{2\Omega} \\ g_u \mathbf{I}_{2\Omega} & g_v \mathbf{I}_{2\Omega} + \mathcal{L}_v + D_v^{12}\mathcal{I} \end{pmatrix}$$

$$\mathcal{L}_u = \begin{pmatrix} D_u^1 \mathbf{L}^1 & \mathbf{0} \\ \mathbf{0} & D_u^2 \mathbf{L}^2 \end{pmatrix}$$

$$\mathcal{L}_u + D_u^{12}\mathcal{I}$$

supra-Laplace matrix

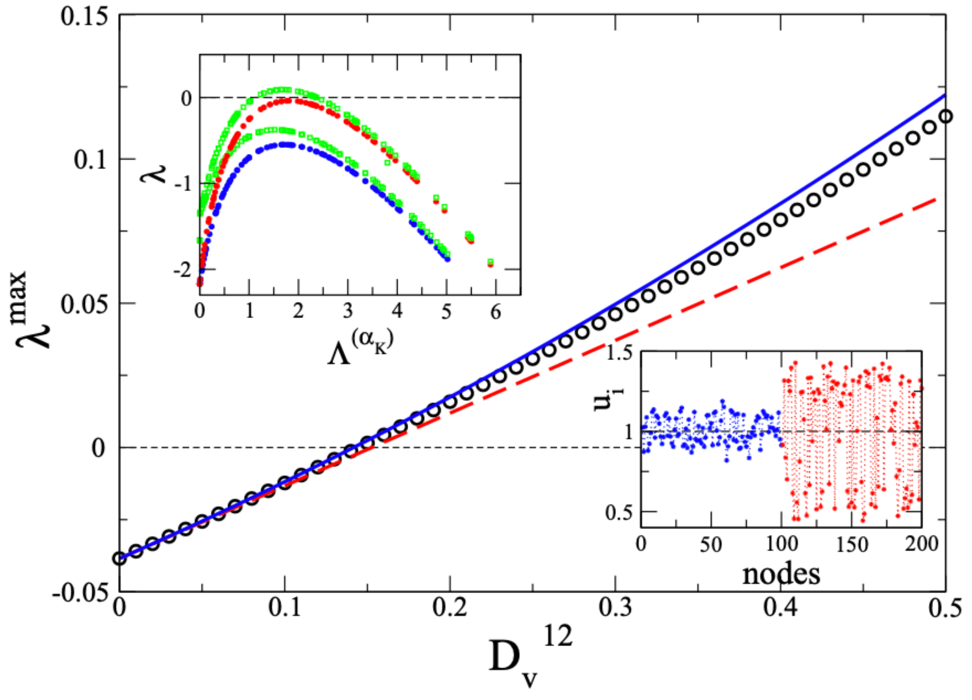
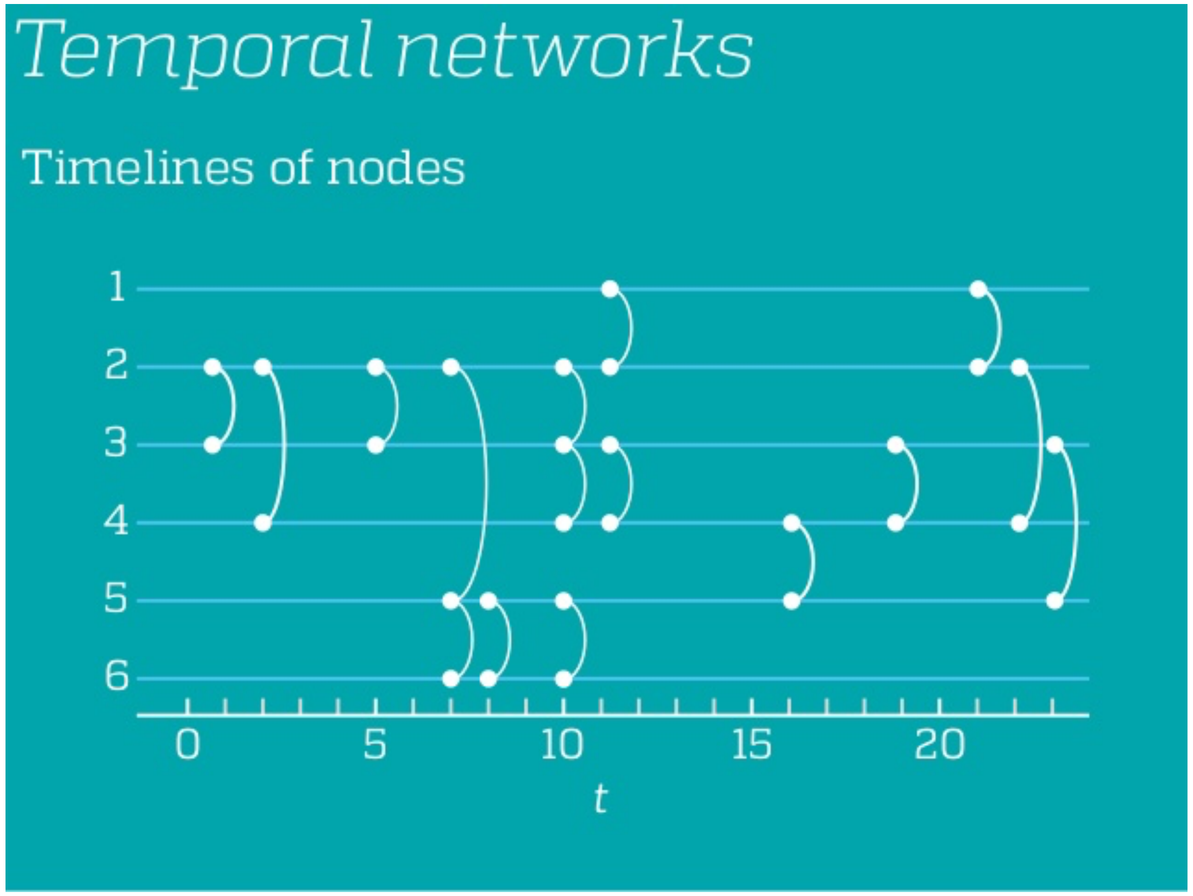
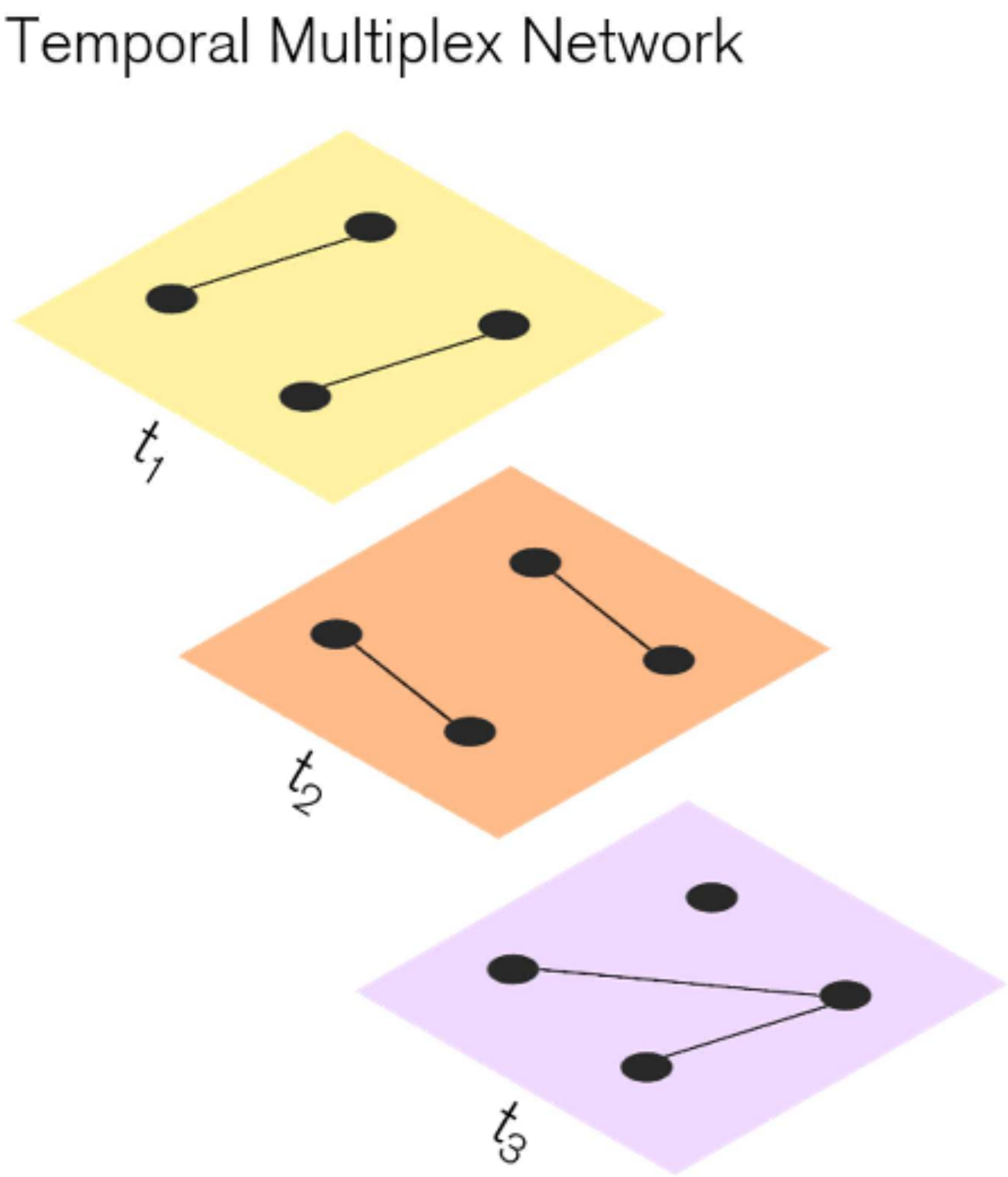


FIG. 2. (Color online) Main: $\lambda^{\max}$ is plotted vs $D_v^{12}$, starting from a condition for which the instability cannot occur when $D_v^{12} = 0$. Circles refer to a direct numerical computation of $\lambda^{\max}$. The dashed (respectively solid) line represents the analytical solution as obtained at the first (respectively second) perturbative order. Upper inset: the dispersion relation $\lambda$ is plotted versus the eigenvalues of the (single layer) Laplacian operators, $L^1$ and $L^2$. The circles (respectively red and blue online) stand for $D_u^{12} = D_v^{12} = 0$, while the squares (green online) are analytically calculated from (5), at the second order, for $D_u^{12} = 0$ and $D_v^{12} = 0.5$. The two layers of the multiplex have been generated as Watts-Strogatz (WD) [23] networks with probability of rewiring $p$ respectively equal to 0.4 and 0.6. The parameters are $b = 8$, $c = 17$, $D_u^1 = D_u^2 = 1$, $D_v^1 = 4$, $D_v^2 = 5$. Lower inset: asymptotic concentration of species $u$ as function of the nodes index $i$. The first (blue online) $\Omega = 100$ nodes refer to the network with $p = 0.4$, the other $\Omega$ (red online) to $p = 0.6$.

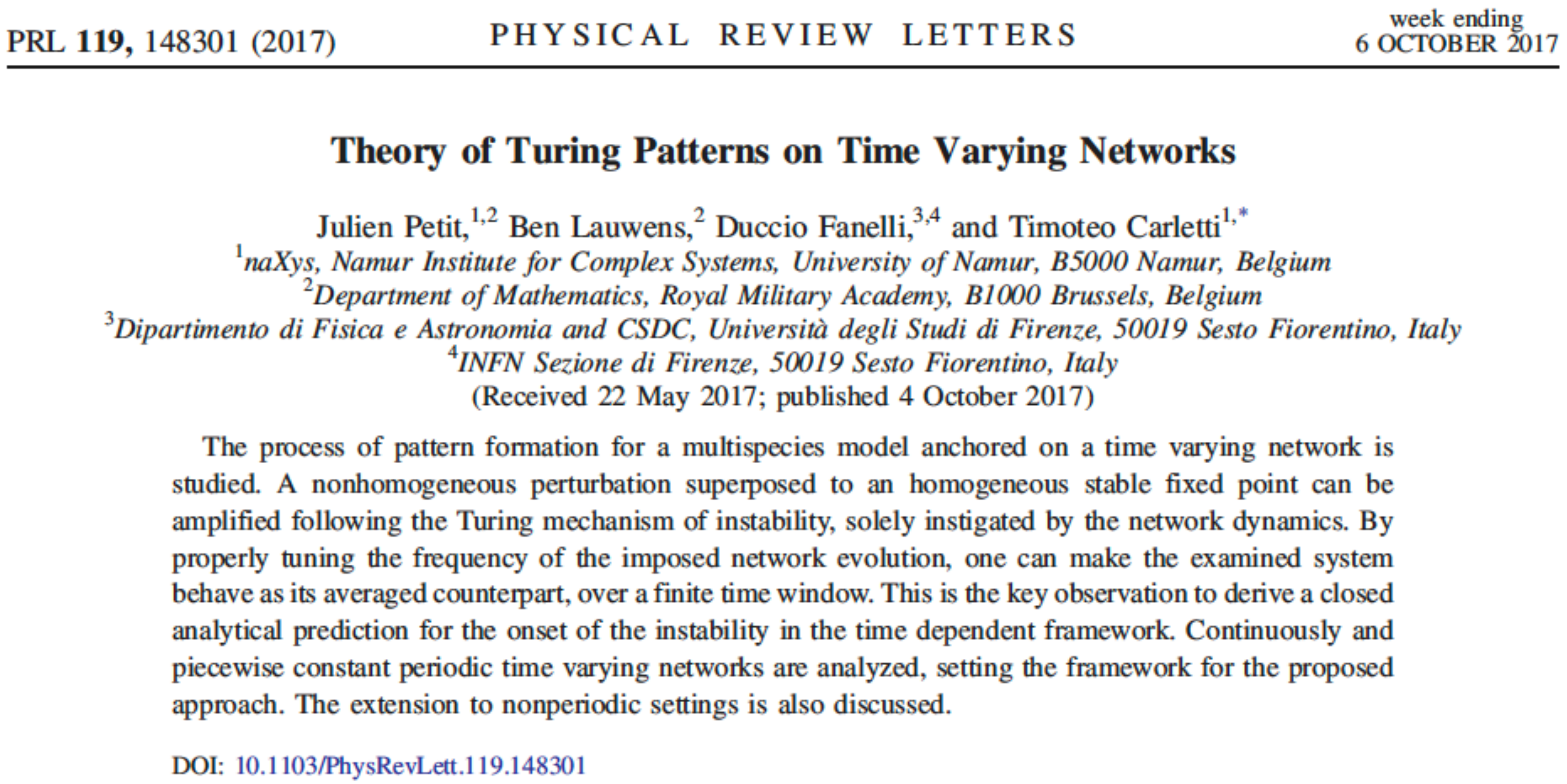
Asllani M et al. 2014 Turing patterns in multiplex networks. Phys Rev E 90, 042814

# Time varying networks




Temporal Multiplex Network

# Time varying networks



PRL **119,** 148301 (2017) PHYSICAL REVIEW LETTERS week ending 6 OCTOBER 2017

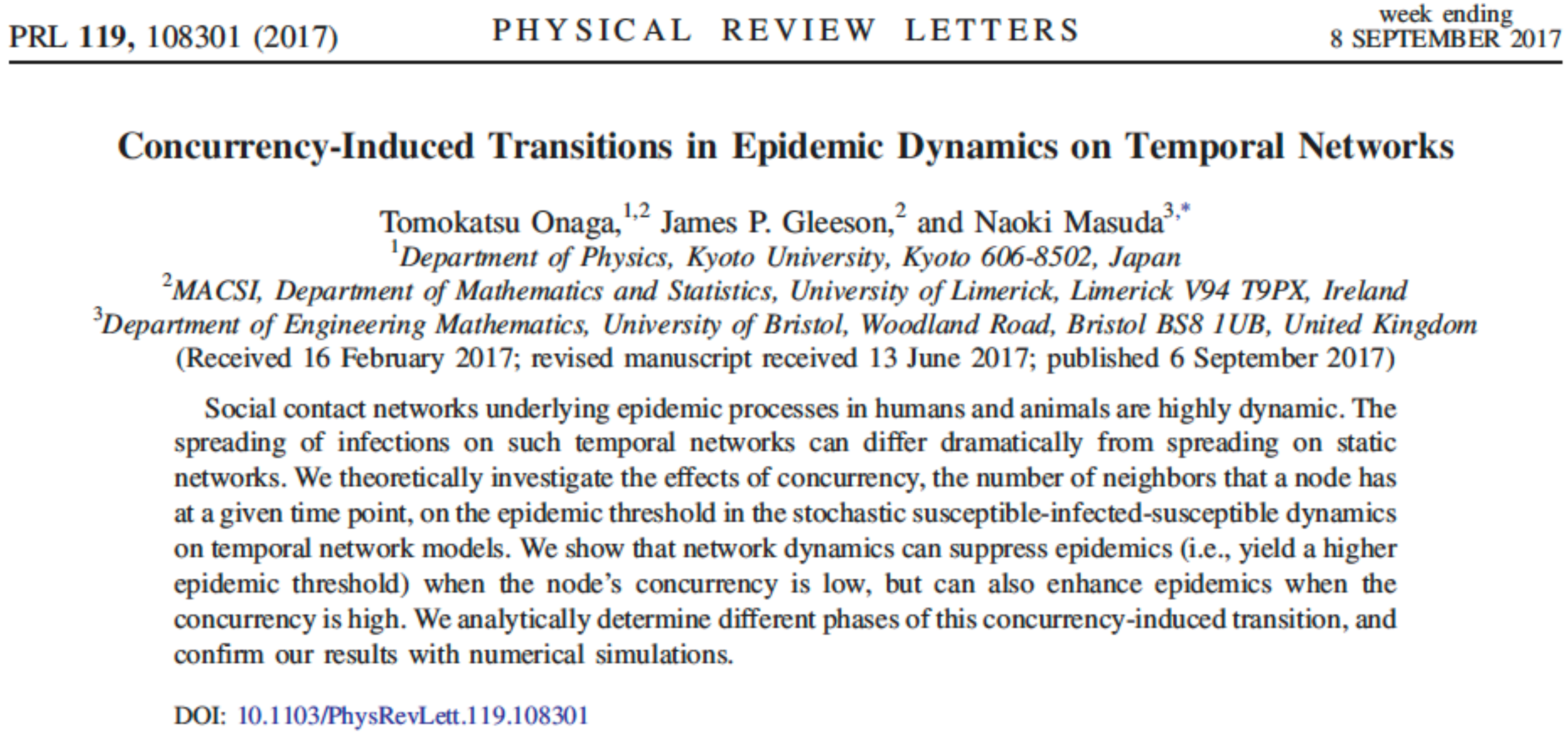
## Theory of Turing Patterns on Time Varying Networks

Julien Petit,[1,2] Ben Lauwens,[2] Duccio Fanelli,[3,4] and Timoteo Carletti[1,*]
[1]*naXys, Namur Institute for Complex Systems, University of Namur, B5000 Namur, Belgium*
[2]*Department of Mathematics, Royal Military Academy, B1000 Brussels, Belgium*
[3]*Dipartimento di Fisica e Astronomia and CSDC, Università degli Studi di Firenze, 50019 Sesto Fiorentino, Italy*
[4]*INFN Sezione di Firenze, 50019 Sesto Fiorentino, Italy*

(Received 22 May 2017; published 4 October 2017)

The process of pattern formation for a multispecies model anchored on a time varying network is studied. A nonhomogeneous perturbation superposed to an homogeneous stable fixed point can be amplified following the Turing mechanism of instability, solely instigated by the network dynamics. By properly tuning the frequency of the imposed network evolution, one can make the examined system behave as its averaged counterpart, over a finite time window. This is the key observation to derive a closed analytical prediction for the onset of the instability in the time dependent framework. Continuously and piecewise constant periodic time varying networks are analyzed, setting the framework for the proposed approach. The extension to nonperiodic settings is also discussed.

DOI: 10.1103/PhysRevLett.119.148301



PRL **119,** 108301 (2017) PHYSICAL REVIEW LETTERS week ending 8 SEPTEMBER 2017

## Concurrency-Induced Transitions in Epidemic Dynamics on Temporal Networks

Tomokatsu Onaga,[1,2] James P. Gleeson,[2] and Naoki Masuda[3,*]
[1]*Department of Physics, Kyoto University, Kyoto 606-8502, Japan*
[2]*MACSI, Department of Mathematics and Statistics, University of Limerick, Limerick V94 T9PX, Ireland*
[3]*Department of Engineering Mathematics, University of Bristol, Woodland Road, Bristol BS8 1UB, United Kingdom*

(Received 16 February 2017; revised manuscript received 13 June 2017; published 6 September 2017)

Social contact networks underlying epidemic processes in humans and animals are highly dynamic. The spreading of infections on such temporal networks can differ dramatically from spreading on static networks. We theoretically investigate the effects of concurrency, the number of neighbors that a node has at a given time point, on the epidemic threshold in the stochastic susceptible-infected-susceptible dynamics on temporal network models. We show that network dynamics can suppress epidemics (i.e., yield a higher epidemic threshold) when the node's concurrency is low, but can also enhance epidemics when the concurrency is high. We analytically determine different phases of this concurrency-induced transition, and confirm our results with numerical simulations.

DOI: 10.1103/PhysRevLett.119.108301

# Time varying networks

$$\dot{u}_i(t) = f(u_i, v_i) + D_u \sum_{j=1}^{N} L_{ij}(t/\epsilon) u_j(t),$$

$$\dot{v}_i(t) = g(u_i, v_i) + D_v \sum_{j=1}^{N} L_{ij}(t/\epsilon) v_j(t),$$

$$\dot{u}_i(t) = f(u_i, v_i) + D_u \sum_{j=1}^{N} \langle L_{ij} \rangle u_j,$$

$$\dot{v}_i(t) = g(u_i, v_i) + D_v \sum_{j=1}^{N} \langle L_{ij} \rangle v_j.$$

## Theory of average

Petit J et al. 2017 Theory of Turing patterns on time varying networks.
Phys Rev L 119, 148301

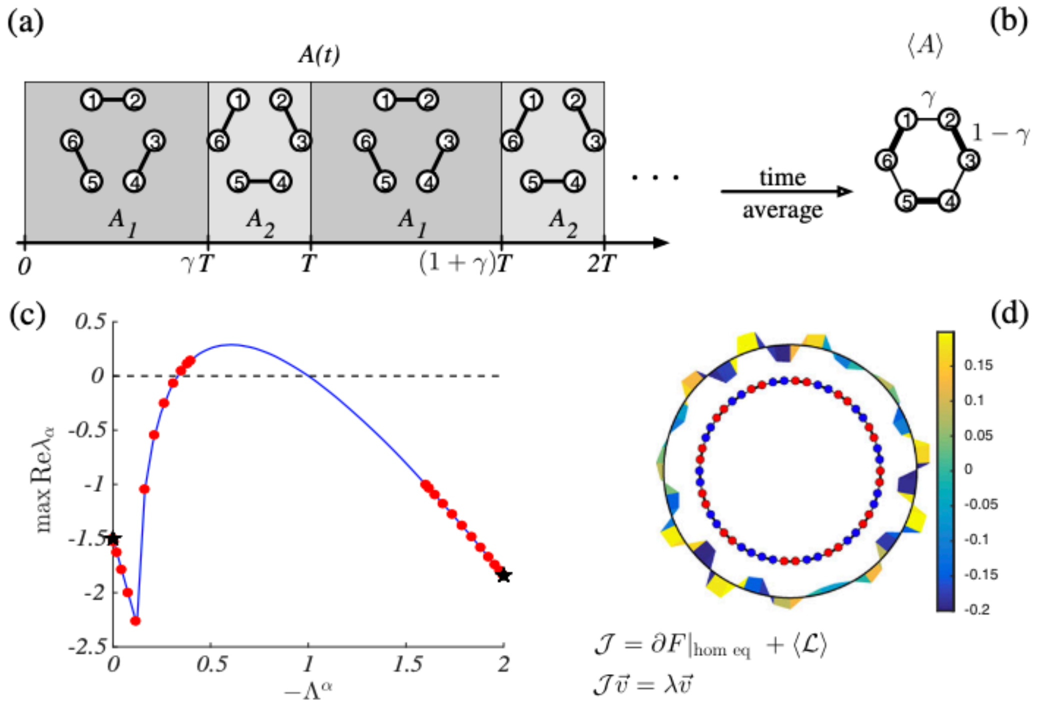
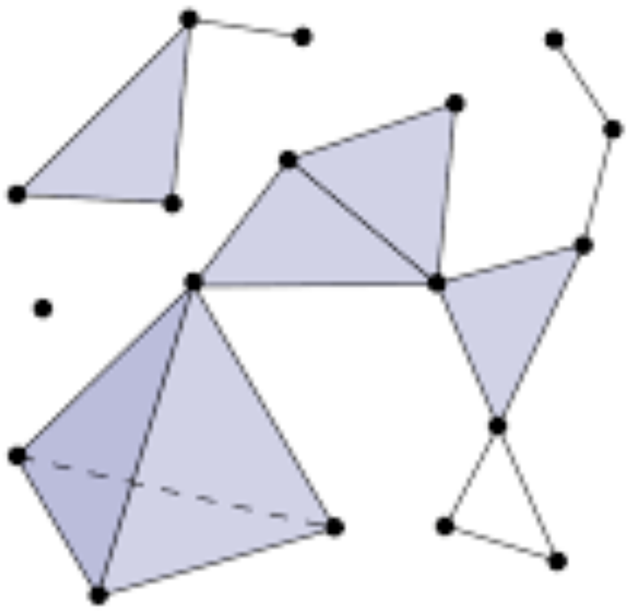


FIG. 1. Twin network. (a) $T$-periodic network built from two static networks, of adjacency matrices $\mathbf{A}_1$ and $\mathbf{A}_2$. Each network in this illustrative example is made of $N = 6$ nodes. In the network stored in matrix $\mathbf{A}_1$, symmetric edges are drawn between the pairs (1,2), (3,4), and (5,6). The second network, embodied in matrix $\mathbf{A}_2$, links nodes (6,1), (2,3), and (4,5). For $t \in [0, \gamma T)$, the $T$-periodic network coincides with $\mathbf{A}_1$, $\mathbf{A}(t) = \mathbf{A}_1$, while, in $[\gamma T, T)$, we set $\mathbf{A}(t) = \mathbf{A}_2$. The time varying network is then obtained by iterating the process in time. (b) The ensuing time averaged network $\langle \mathbf{A} \rangle = \gamma \mathbf{A}_1 + (1-\gamma)\mathbf{A}_2$. (c) Dispersion relation (max Re$\lambda_\alpha$ vs $-\Lambda_\alpha$) or the averaged network (the red circles), for each static twin network (the black stars) and for the continuous support case (the blue curve). Here, the networks are generated as discussed above, but now $N = 50$. (d) Patterns in the averaged network. Nodes are blue if they present an excess of concentration with respect to the homogeneous equilibrium solution ($[u_i(\infty) - \bar{u}] \geq 0.1$) and red otherwise ($[u_i(\infty) - \bar{u}] \leq -0.1$). The outer drawing represents the entries of $\vec{v}$, the eigenvector of the Jacobian matrix $\mathcal{J}$ associated with the eigenvalues that yields the largest value of the dispersion relation. The black ring represents the zeroth level; red and yellow areas are associated with positive entries of $\vec{v}$, while blue and light blue regions refer to negative values. The reaction model is the Brusselator with $b = 8$, $c = 10$, $D_u = 3$, and $D_v = 10$. The homogeneous equilibrium is $\bar{u} = 1$ and $\bar{v} = 0.8$. The remaining parameters are set to $\gamma = 0.3$, $T = 1$, $D_u = 3$, and $D_v = 10$.

# Simplicial complexes

## Simplicial complexes

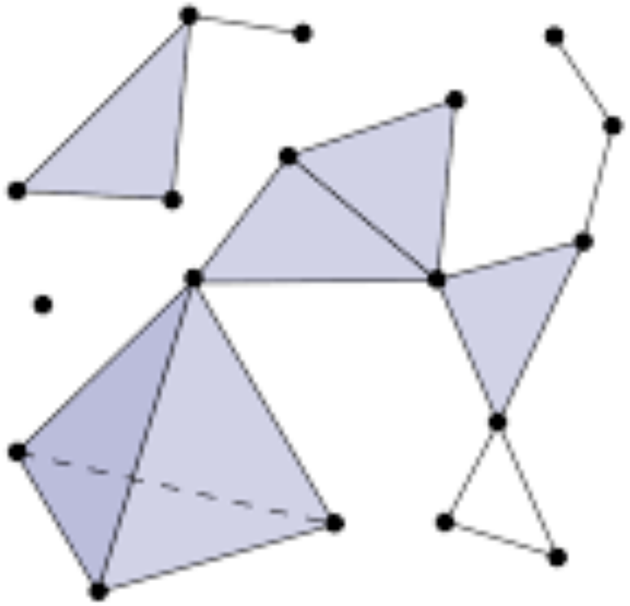


d-simplex = d+1 nodes
(all linked together)

1-simplex = link
2-simplex = triangle
3-simplex = tetrahedron

# Simplicial complexes and Hypergraphs

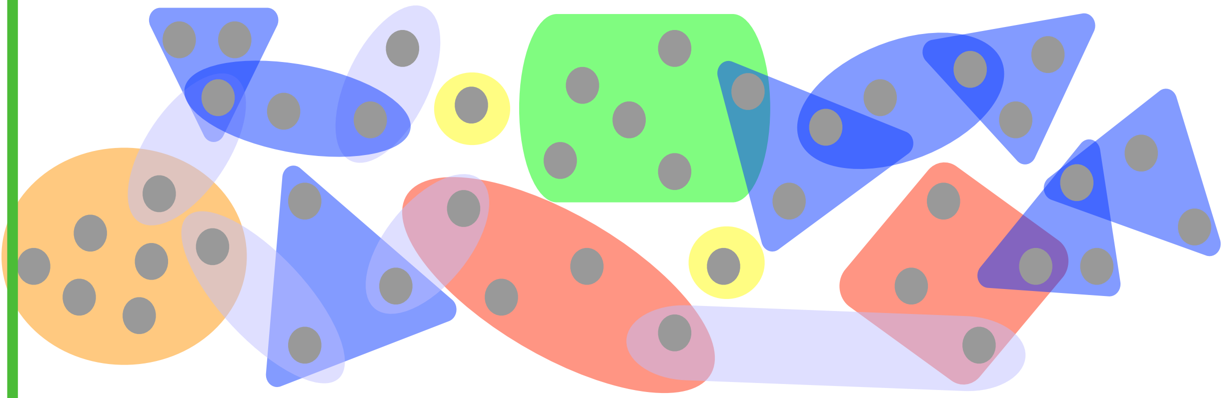
## Simplicial complexes



d-simplex = d+1 nodes
(all linked together)

1-simplex = link
2-simplex = triangle
3-simplex = tetrahedron

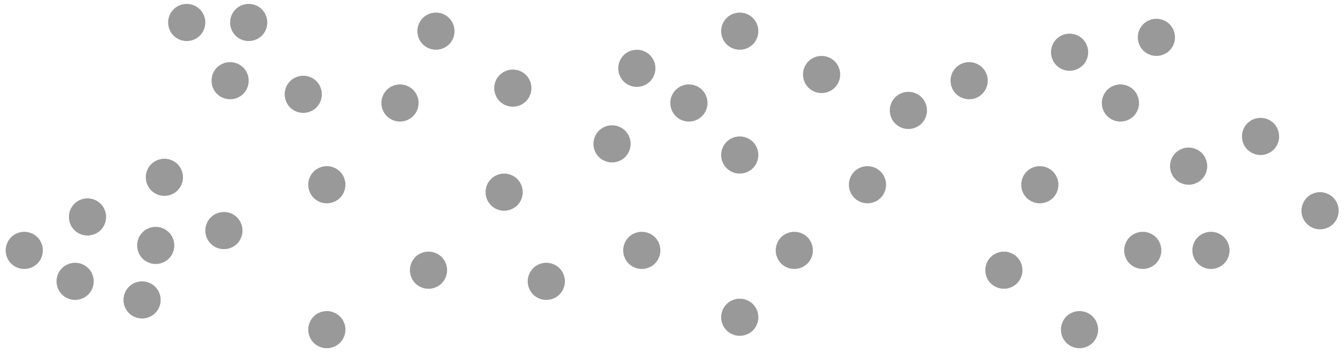
## Hypergraphs



hyperedge = set of nodes

timoteo.carletti@unamur.be

# Hypergraphs

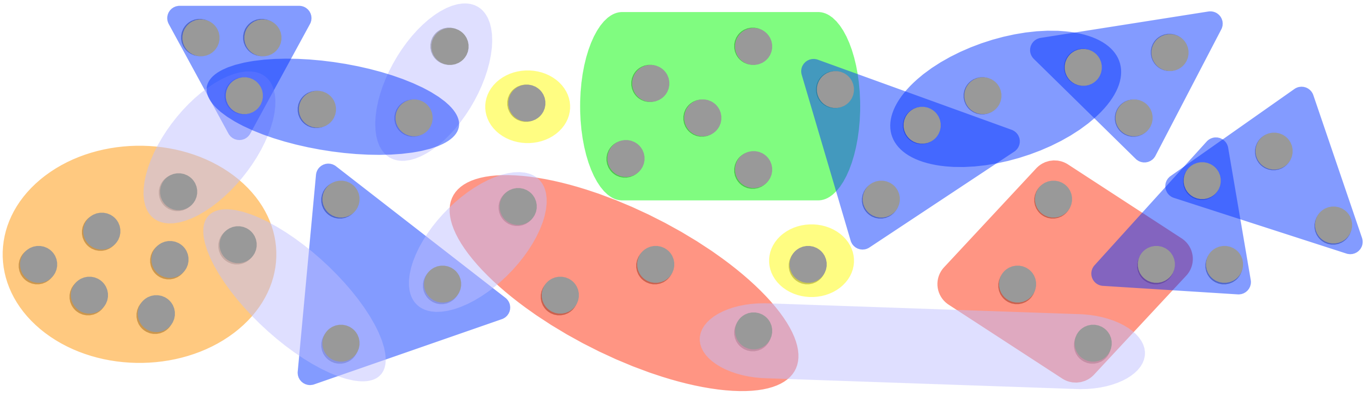


ensemble of nodes

# Hypergraphs

$|E_{\alpha_3}| = 1$

$|E_{\alpha_2}| = 3$



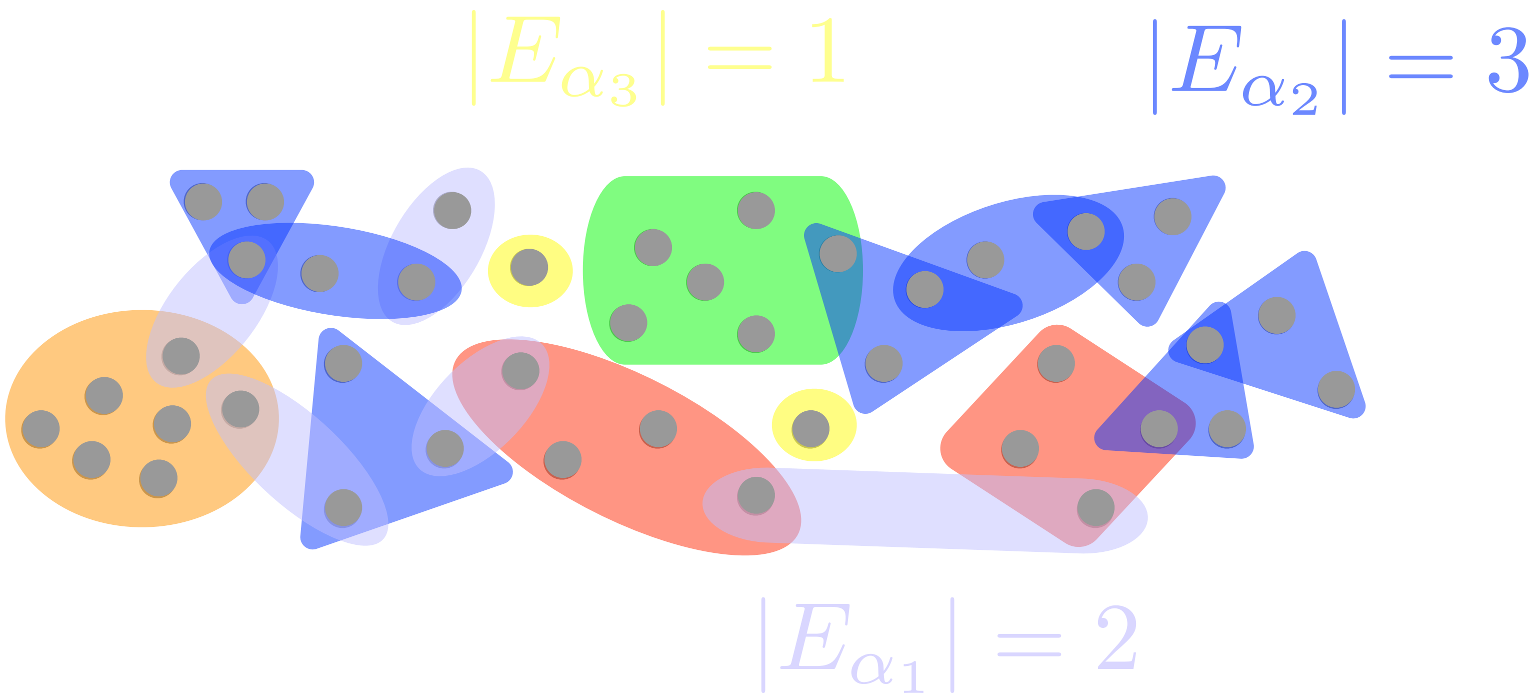
$|E_{\alpha_1}| = 2$

ensemble of nodes

hyperedges = set of nodes

timoteo.carletti@unamur.be

# Hypergraphs



ensemble of nodes

hyperedges

Incidence matrix

$e_{i\alpha} = 1 \quad \text{iff } i \in E_\alpha$

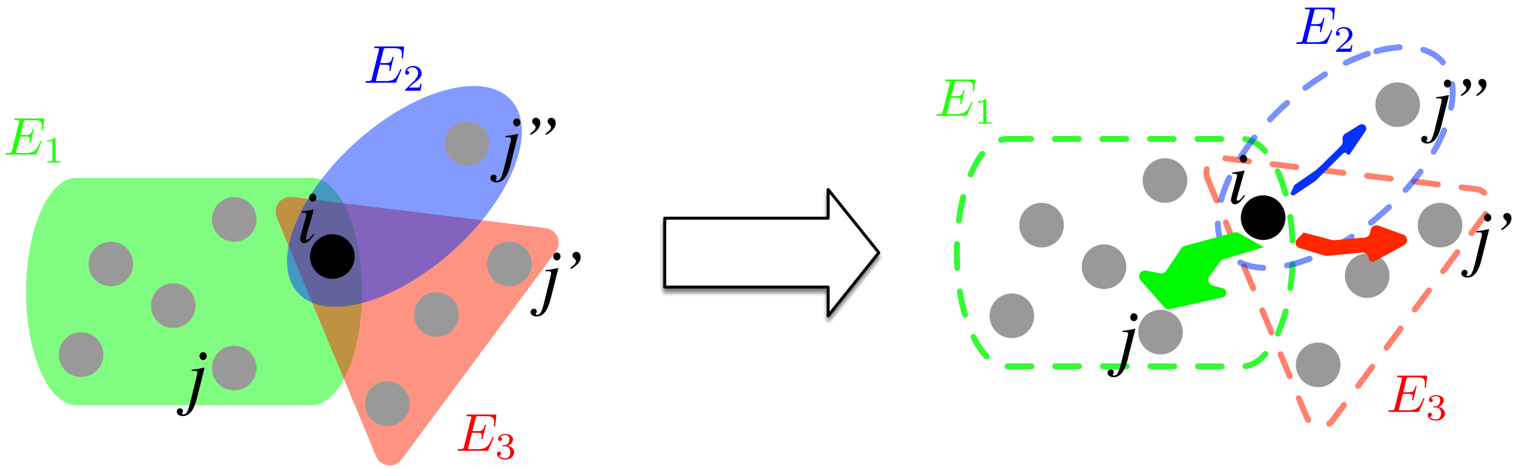
Hyperadjacency matrix

$A = ee^{\mathrm{T}}$

Hyperedge matrix

$C = e^{\mathrm{T}}e$

# Hypergraphs



$$k_{ij}^H = \sum_\alpha (C_{\alpha\alpha} - 1)^\tau e_{i\alpha} e_{j\alpha}$$

hyperedge size

incidence matrices

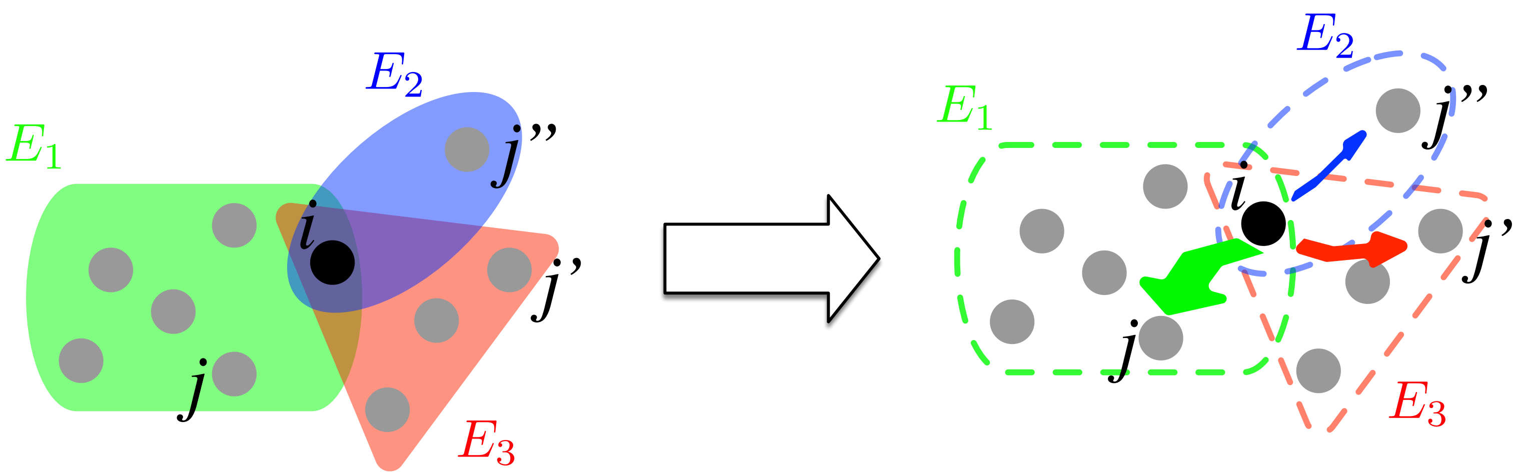
PHYSICAL REVIEW E **101**, 022308 (2020)

**Random walks on hypergraphs**

Timoteo Carletti,[1] Federico Battiston,[2] Giulia Cencetti,[3] and Duccio Fanelli[4]
[1]*Namur Institute for Complex Systems, University of Namur, 5000 Namur, Belgium*
[2]*Department of Network and Data Science, Central European University, Budapest 1051, Hungary*
[3]*Mobile and Social Computing Lab, Fondazione Bruno Kessler, Via Sommarive 18, 38123 Povo, Trento, Italy*
[4]*Dipartimento di Fisica e Astronomia, Università di Firenze, INFN, and CSDC, Via Sansone 1, 50019 Sesto Fiorentino, Firenze, Italy*

(Received 14 November 2019; accepted 20 January 2020; published 18 February 2020)

# Hypergraphs



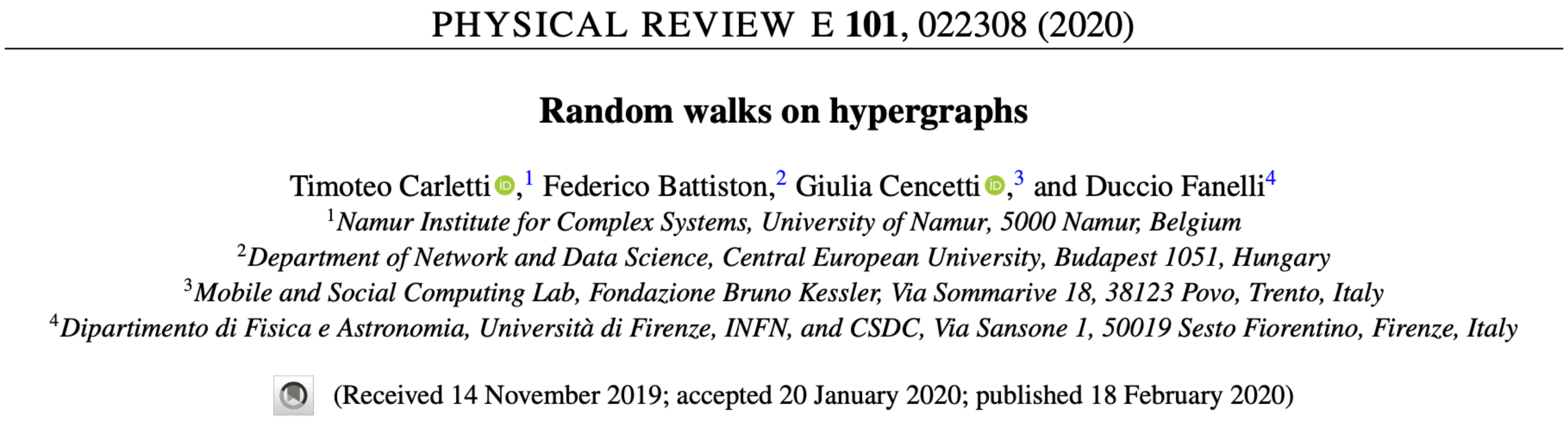
$$\boxed{L^H_{ij} = \delta_{ij} - \frac{k^H_{ij}}{\sum_{\ell\neq i} k^H_{i\ell}}}$$

$$k^H_{ij} = \sum_\alpha (C_{\alpha\alpha} - 1)^\tau e_{i\alpha} e_{j\alpha}$$

hyperedge size

incidence matrices

$$p_i^{(\infty)} = \frac{k^H_i}{\sum_\ell k^H_\ell}$$



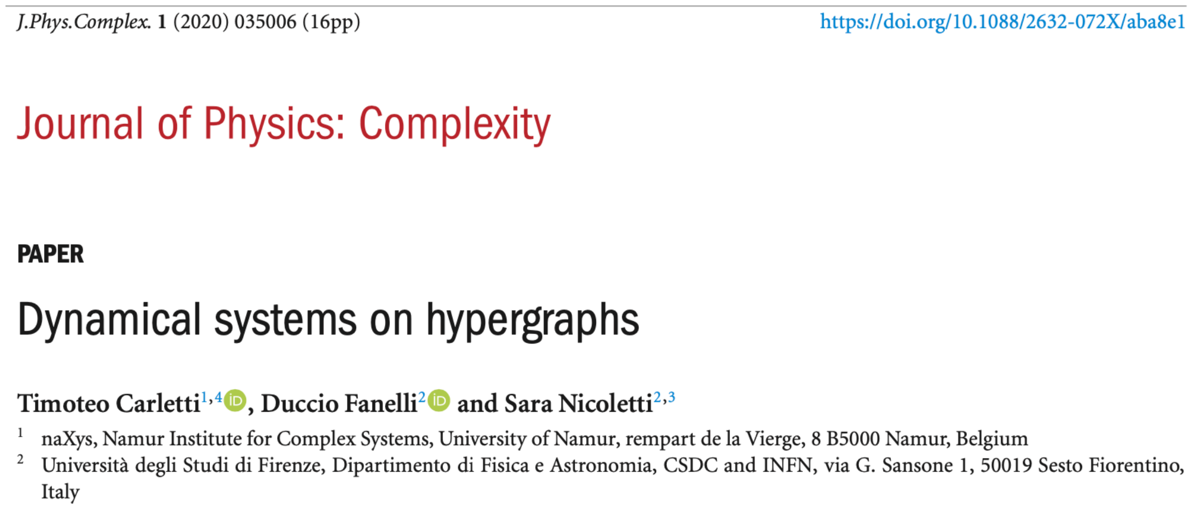
PHYSICAL REVIEW E **101**, 022308 (2020)

## Random walks on hypergraphs

Timoteo Carletti,[1] Federico Battiston,[2] Giulia Cencetti,[3] and Duccio Fanelli[4]
[1] *Namur Institute for Complex Systems, University of Namur, 5000 Namur, Belgium*
[2] *Department of Network and Data Science, Central European University, Budapest 1051, Hungary*
[3] *Mobile and Social Computing Lab, Fondazione Bruno Kessler, Via Sommarive 18, 38123 Povo, Trento, Italy*
[4] *Dipartimento di Fisica e Astronomia, Università di Firenze, INFN, and CSDC, Via Sansone 1, 50019 Sesto Fiorentino, Firenze, Italy*

(Received 14 November 2019; accepted 20 January 2020; published 18 February 2020)

# Hypergraphs



J.Phys.Complex. **1** (2020) 035006 (16pp) https://doi.org/10.1088/2632-072X/aba8e1

Journal of Physics: Complexity

**PAPER**

## Dynamical systems on hypergraphs

**Timoteo Carletti**[1,4], **Duccio Fanelli**[2] **and Sara Nicoletti**[2,3]

[1] naXys, Namur Institute for Complex Systems, University of Namur, rempart de la Vierge, 8 B5000 Namur, Belgium
[2] Università degli Studi di Firenze, Dipartimento di Fisica e Astronomia, CSDC and INFN, via G. Sansone 1, 50019 Sesto Fiorentino, Italy
[3] Dipartimento di Ingegneria dell'Informazione, Università di Firenze, V
[4] Author to whom any correspondence should be addressed.

**E-mail:** timoteo.carletti@unamur.be

**Keywords:** hypergraphs, master stability function, synchronisation, Turir

$$
\begin{aligned}
\frac{d\mathbf{x}_i}{dt} &= \mathbf{F}(\mathbf{x}_i) - \varepsilon \sum_{\alpha,j} e_{i\alpha} e_{j\alpha} (C_{\alpha\,\alpha} - 1) \left(\mathbf{G}(\mathbf{x}_i) - \mathbf{G}(\mathbf{x}_j)\right) \\
&= \mathbf{F}(\mathbf{x}_i) - \varepsilon \sum_{j} k^H_{ij} \left(\mathbf{G}(\mathbf{x}_i) - \mathbf{G}(\mathbf{x}_j)\right) = \mathbf{F}(\mathbf{x}_i) - \varepsilon \sum_{j} \left(\delta_{ij} k^H_i - k^H_{ij}\right) \mathbf{G}(\mathbf{x}_j) \\
&= \mathbf{F}(\mathbf{x}_i) - \varepsilon \sum_{j} L^H_{ij} \mathbf{G}(\mathbf{x}_j),
\end{aligned}
$$

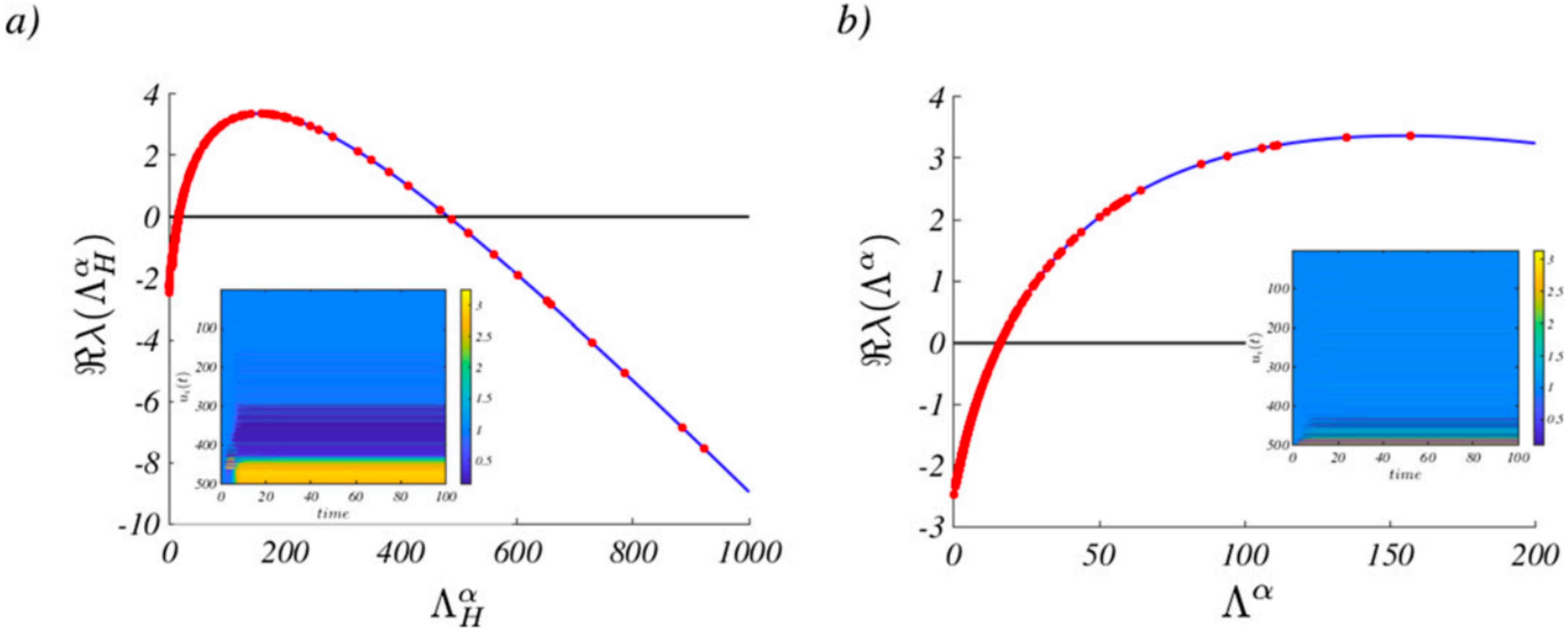
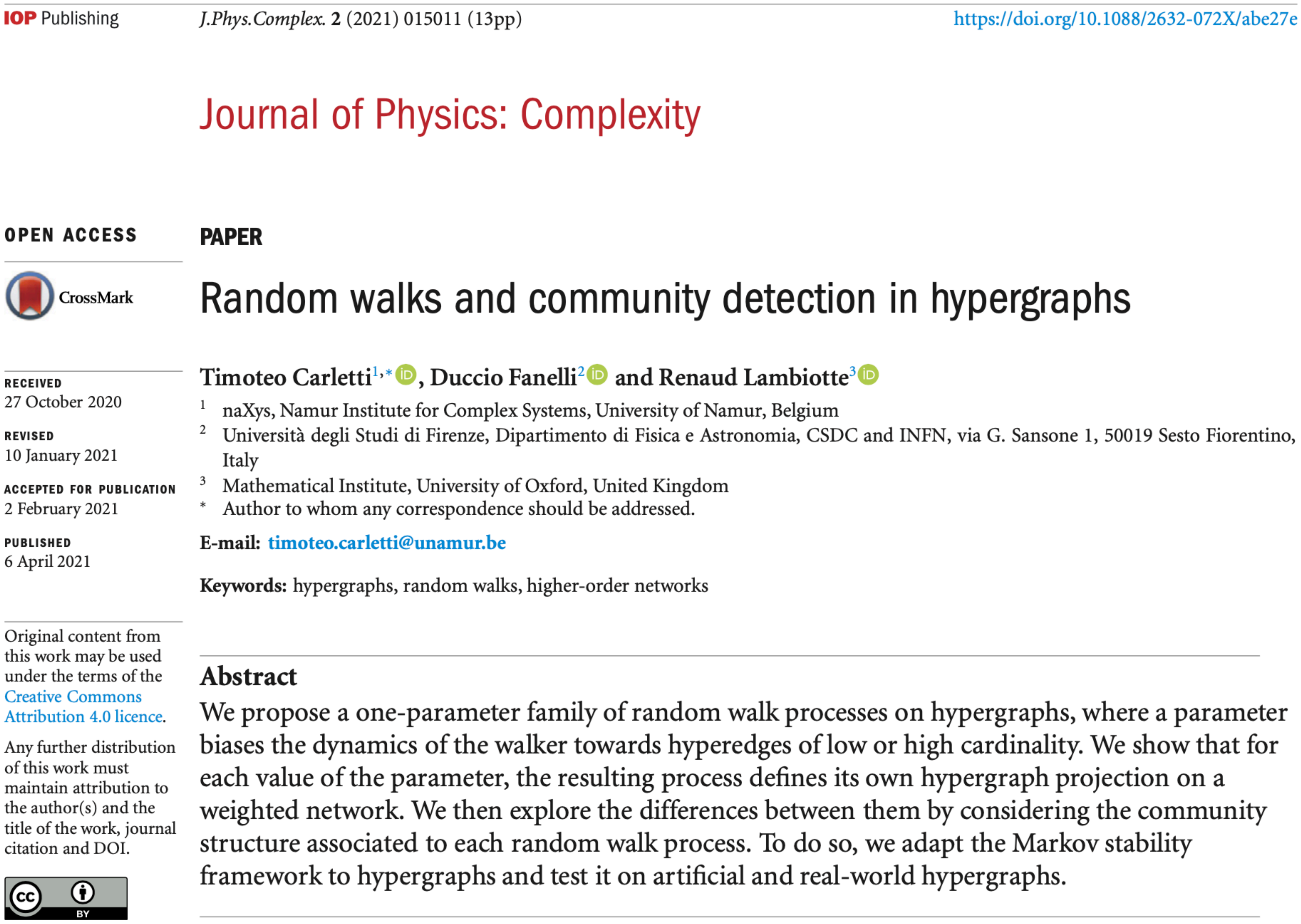


**Figure 5.** Turing patterns on hypergraphs. Main panels: the dispersion relation for the Brusselator model defined on the hypergraph—panel (a)—and the projected network—panel (b). One can observe that in both cases there are eigenvalues for which the dispersion relation is positive (red dots); the blue line represents the dispersion relation for the Brusselator model defined on a continuous regular support. Being both Laplace matrices symmetric, the dispersion relation computed for the discrete spectra lies on top of the one obtained for the continuous support. Insets: the Turing patterns on the hypergraph (panel (a)) and the projected network (panel (b)). We report the time evolution of the concentration of the species $u_i(t)$ in each node as a function of time, by using an appropriate colour code (yellow associated to large values, blue to small ones). In the former case, nodes are ordered for increasing hyper degree while in the second panel for increasing degree. One can hence conclude that nodes associated to large hyper degrees display a large concentration amount for species $u_i$. This yields a very localised pattern. The hypergraph and the projected network are the same used in figure 2.

# Hypergraphs



IOP Publishing J.Phys.Complex. **2** (2021) 015011 (13pp) https://doi.org/10.1088/2632-072X/abe27e

Journal of Physics: Complexity

OPEN ACCESS

CrossMark

RECEIVED
27 October 2020

REVISED
10 January 2021

ACCEPTED FOR PUBLICATION
2 February 2021

PUBLISHED
6 April 2021

**PAPER**

## Random walks and community detection in hypergraphs

**Timoteo Carletti**[1,*], **Duccio Fanelli**[2] and **Renaud Lambiotte**[3]

[1] naXys, Namur Institute for Complex Systems, University of Namur, Belgium
[2] Università degli Studi di Firenze, Dipartimento di Fisica e Astronomia, CSDC and INFN, via G. Sansone 1, 50019 Sesto Fiorentino, Italy
[3] Mathematical Institute, University of Oxford, United Kingdom
[*] Author to whom any correspondence should be addressed.

**E-mail:** timoteo.carletti@unamur.be

**Keywords:** hypergraphs, random walks, higher-order networks

**Abstract**

We propose a one-parameter family of random walk processes on hypergraphs, where a parameter biases the dynamics of the walker towards hyperedges of low or high cardinality. We show that for each value of the parameter, the resulting process defines its own hypergraph projection on a weighted network. We then explore the differences between them by considering the community structure associated to each random walk process. To do so, we adapt the Markov stability framework to hypergraphs and test it on artificial and real-world hypergraphs.

# I didn't have time but ...

The European Physical Journal

volume 89 · number 12 · december · 2016

EPJ B

Recognized by European Physical Society

Condensed Matter and Complex Systems

From:
Tune the topology
to create or destroy patterns

by M. Asllani et al.

edp sciences

Società Italiana di Fisica

Springer

Eur. Phys. J. B (2016) 89: 260
DOI: 10.1140/epjb/e2016-70248-6

THE EUROPEAN
PHYSICAL JOURNAL B

Regular Article

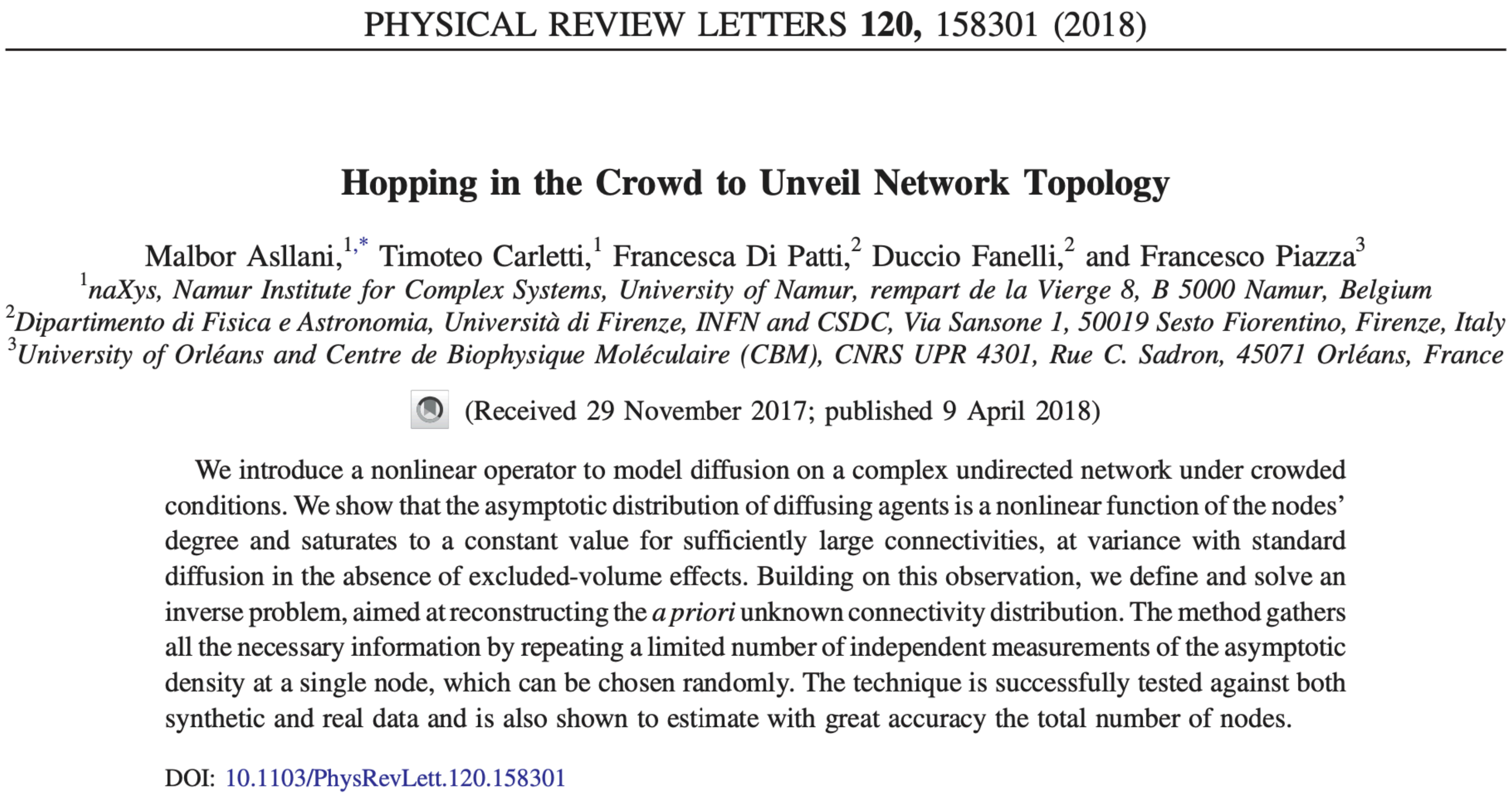
## Tune the topology to create or destroy patterns

Malbor Asllani[1,a], Timoteo Carletti[1], and Duccio Fanelli[2]

[1] naXys, Namur Center for Complex Systems, University of Namur, rempart de la Vierge 8, B 5000 Namur, Belgium
[2] Dipartimento di Fisica e Astronomia, University of Florence, INFN and CSDC, via Sansone 1, 50019 Sesto Fiorentino, Florence, Italy

Received 24 April 2016 / Received in final form 5 September 2016

# I didn’t have time but …



PHYSICAL REVIEW LETTERS **120,** 158301 (2018)

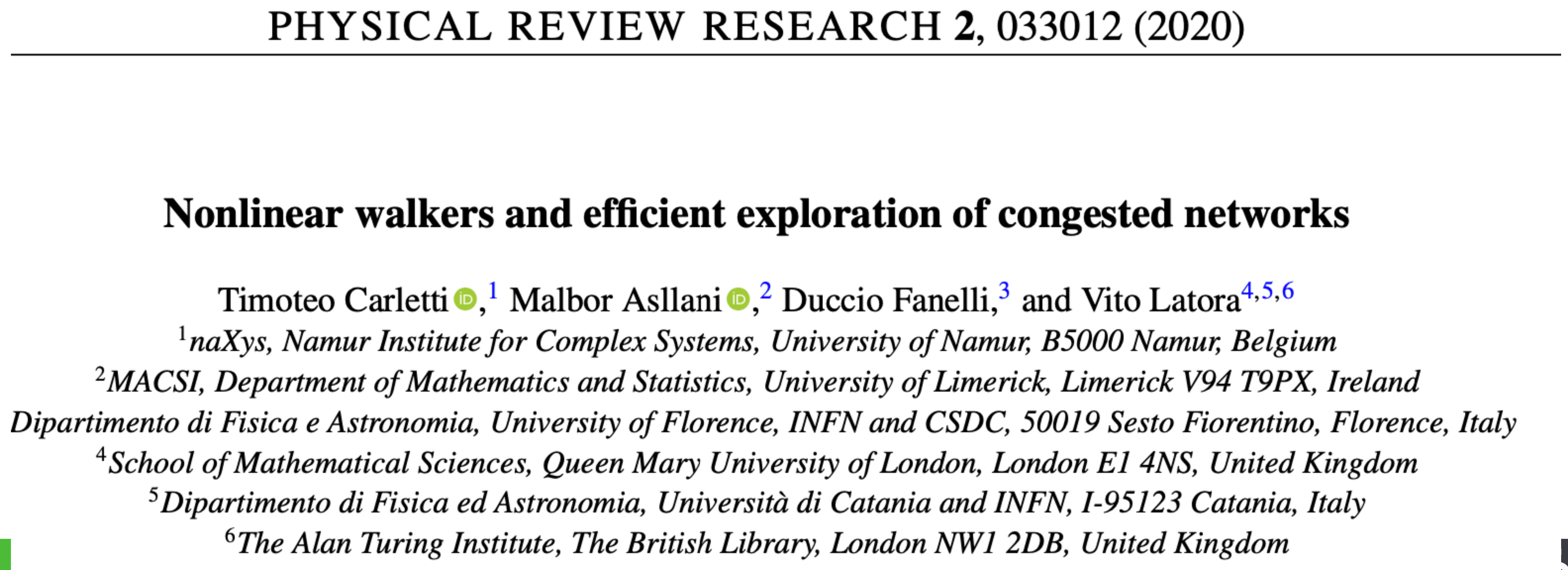
## Hopping in the Crowd to Unveil Network Topology

Malbor Asllani,[1,*] Timoteo Carletti,[1] Francesca Di Patti,[2] Duccio Fanelli,[2] and Francesco Piazza[3]
[1] *naXys, Namur Institute for Complex Systems, University of Namur, rempart de la Vierge 8, B 5000 Namur, Belgium*
[2] *Dipartimento di Fisica e Astronomia, Università di Firenze, INFN and CSDC, Via Sansone 1, 50019 Sesto Fiorentino, Firenze, Italy*
[3] *University of Orléans and Centre de Biophysique Moléculaire (CBM), CNRS UPR 4301, Rue C. Sadron, 45071 Orléans, France*

(Received 29 November 2017; published 9 April 2018)

We introduce a nonlinear operator to model diffusion on a complex undirected network under crowded conditions. We show that the asymptotic distribution of diffusing agents is a nonlinear function of the nodes’ degree and saturates to a constant value for sufficiently large connectivities, at variance with standard diffusion in the absence of excluded-volume effects. Building on this observation, we define and solve an inverse problem, aimed at reconstructing the *a priori* unknown connectivity distribution. The method gathers all the necessary information by repeating a limited number of independent measurements of the asymptotic density at a single node, which can be chosen randomly. The technique is successfully tested against both synthetic and real data and is also shown to estimate with great accuracy the total number of nodes.

DOI: 10.1103/PhysRevLett.120.158301

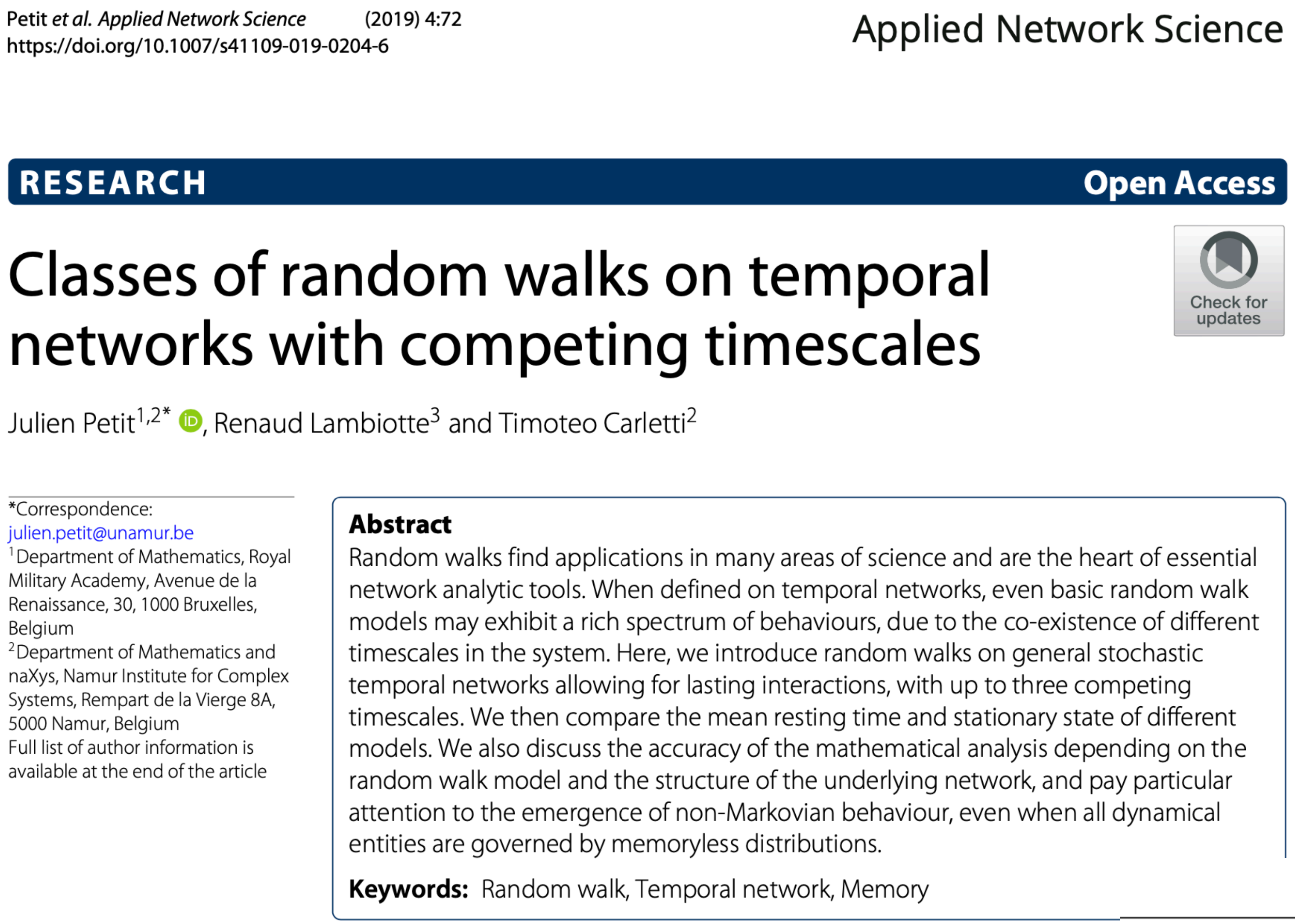


PHYSICAL REVIEW RESEARCH **2**, 033012 (2020)

## Nonlinear walkers and efficient exploration of congested networks

Timoteo Carletti,[1] Malbor Asllani,[2] Duccio Fanelli,[3] and Vito Latora[4,5,6]
[1] *naXys, Namur Institute for Complex Systems, University of Namur, B5000 Namur, Belgium*
[2] *MACSI, Department of Mathematics and Statistics, University of Limerick, Limerick V94 T9PX, Ireland*
*Dipartimento di Fisica e Astronomia, University of Florence, INFN and CSDC, 50019 Sesto Fiorentino, Florence, Italy*
[4] *School of Mathematical Sciences, Queen Mary University of London, London E1 4NS, United Kingdom*
[5] *Dipartimento di Fisica ed Astronomia, Università di Catania and INFN, I-95123 Catania, Italy*
[6] *The Alan Turing Institute, The British Library, London NW1 2DB, United Kingdom*

# I didn't have time but …



Petit *et al. Applied Network Science* (2019) 4:72
https://doi.org/10.1007/s41109-019-0204-6

Applied Network Science

**RESEARCH** **Open Access**

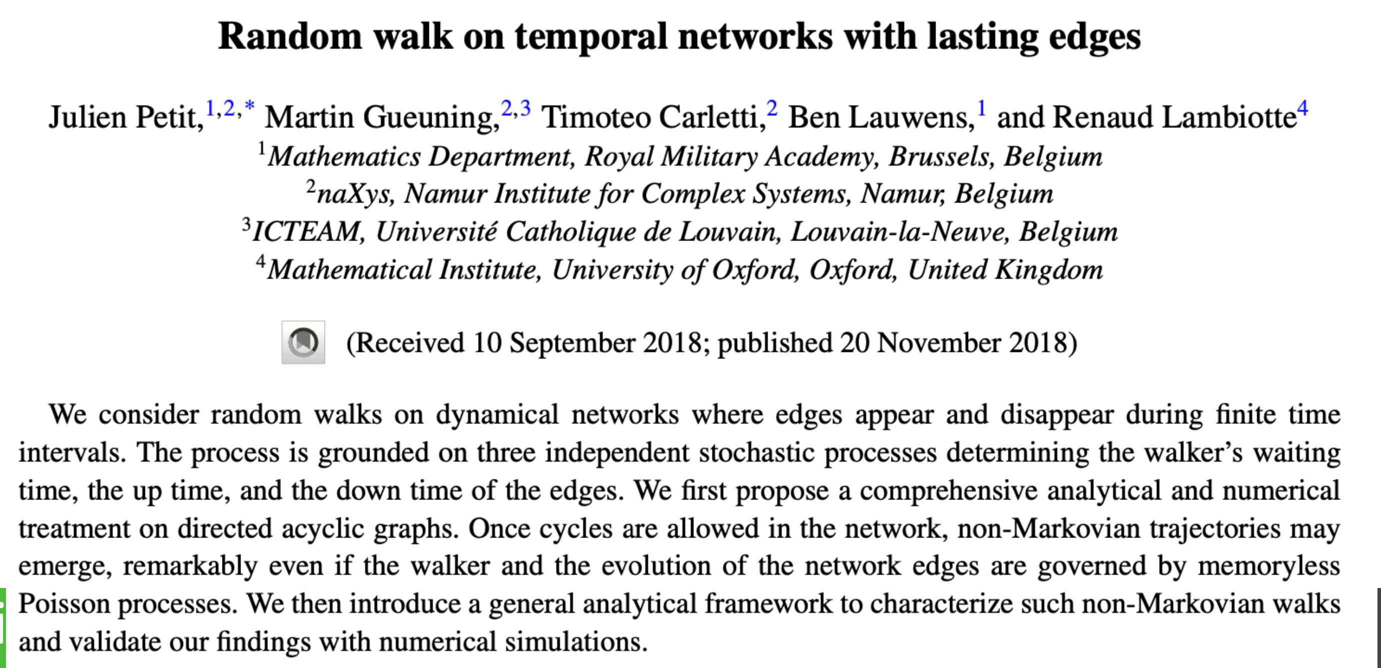
## Classes of random walks on temporal networks with competing timescales

Julien Petit[1,2*], Renaud Lambiotte[3] and Timoteo Carletti[2]

*Correspondence:
julien.petit@unamur.be
[1] Department of Mathematics, Royal Military Academy, Avenue de la Renaissance, 30, 1000 Bruxelles, Belgium
[2] Department of Mathematics and naXys, Namur Institute for Complex Systems, Rempart de la Vierge 8A, 5000 Namur, Belgium
Full list of author information is available at the end of the article

**Abstract**
Random walks find applications in many areas of science and are the heart of essential network analytic tools. When defined on temporal networks, even basic random walk models may exhibit a rich spectrum of behaviours, due to the co-existence of different timescales in the system. Here, we introduce random walks on general stochastic temporal networks allowing for lasting interactions, with up to three competing timescales. We then compare the mean resting time and stationary state of different models. We also discuss the accuracy of the mathematical analysis depending on the random walk model and the structure of the underlying network, and pay particular attention to the emergence of non-Markovian behaviour, even when all dynamical entities are governed by memoryless distributions.

**Keywords:** Random walk, Temporal network, Memory



PHYSICAL REVIEW E **98**, 052307 (2018)

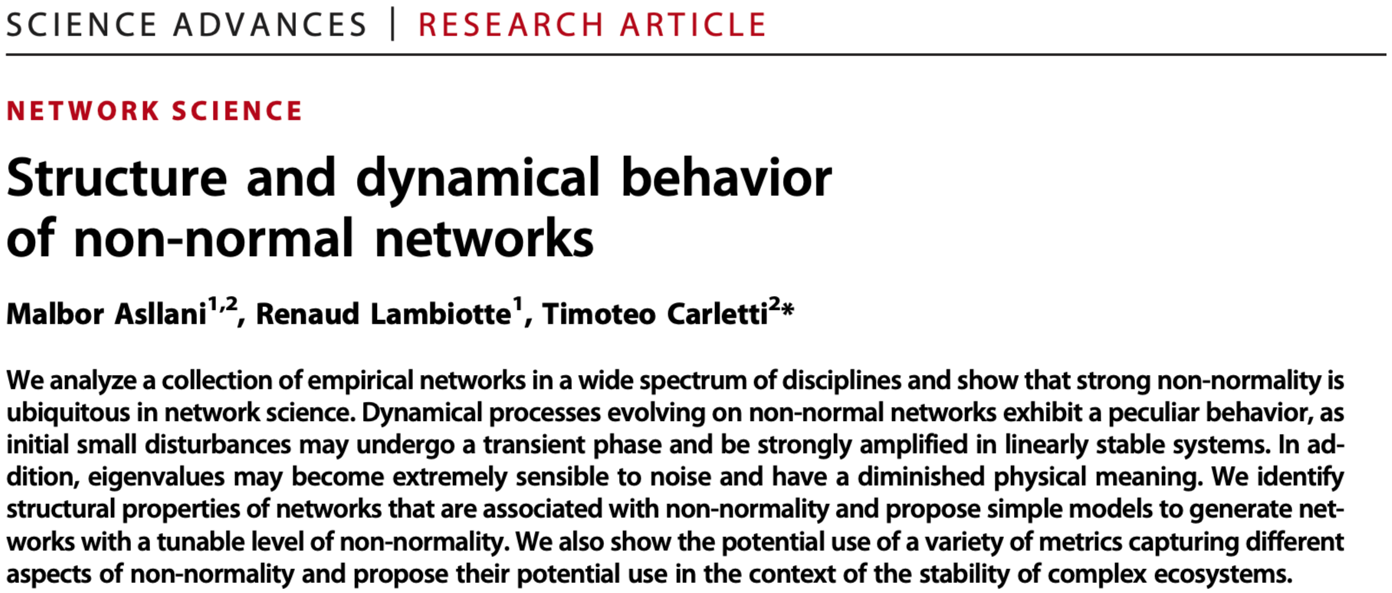
## Random walk on temporal networks with lasting edges

Julien Petit,[1,2,*] Martin Gueuning,[2,3] Timoteo Carletti,[2] Ben Lauwens,[1] and Renaud Lambiotte[4]
[1]*Mathematics Department, Royal Military Academy, Brussels, Belgium*
[2]*naXys, Namur Institute for Complex Systems, Namur, Belgium*
[3]*ICTEAM, Université Catholique de Louvain, Louvain-la-Neuve, Belgium*
[4]*Mathematical Institute, University of Oxford, Oxford, United Kingdom*

(Received 10 September 2018; published 20 November 2018)

We consider random walks on dynamical networks where edges appear and disappear during finite time intervals. The process is grounded on three independent stochastic processes determining the walker's waiting time, the up time, and the down time of the edges. We first propose a comprehensive analytical and numerical treatment on directed acyclic graphs. Once cycles are allowed in the network, non-Markovian trajectories may emerge, remarkably even if the walker and the evolution of the network edges are governed by memoryless Poisson processes. We then introduce a general analytical framework to characterize such non-Markovian walks and validate our findings with numerical simulations.

# I didn't have time but ...



SCIENCE ADVANCES | RESEARCH ARTICLE

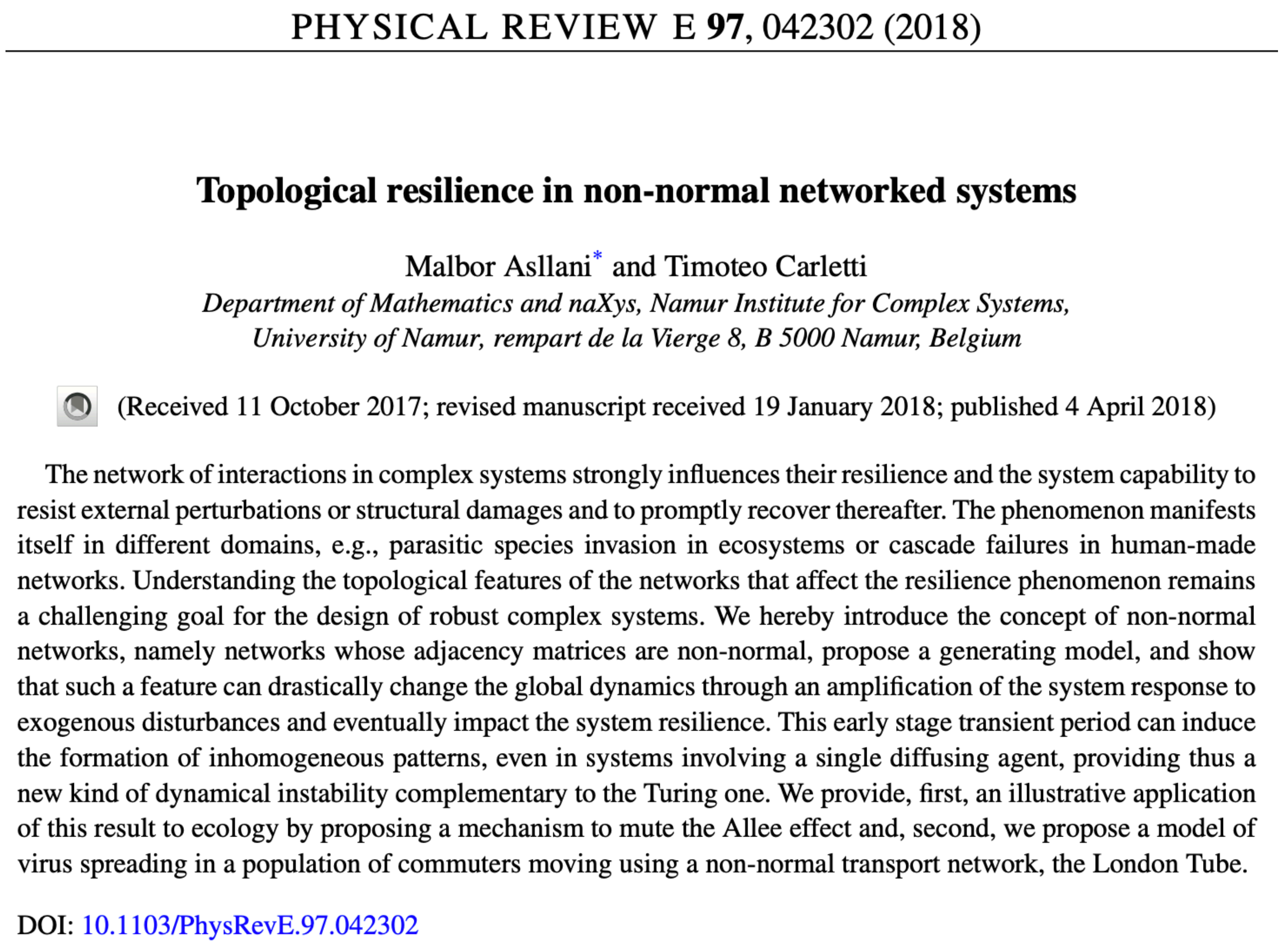
NETWORK SCIENCE

## Structure and dynamical behavior of non-normal networks

**Malbor Asllani[1,2], Renaud Lambiotte[1], Timoteo Carletti[2]***

**We analyze a collection of empirical networks in a wide spectrum of disciplines and show that strong non-normality is ubiquitous in network science. Dynamical processes evolving on non-normal networks exhibit a peculiar behavior, as initial small disturbances may undergo a transient phase and be strongly amplified in linearly stable systems. In addition, eigenvalues may become extremely sensible to noise and have a diminished physical meaning. We identify structural properties of networks that are associated with non-normality and propose simple models to generate networks with a tunable level of non-normality. We also show the potential use of a variety of metrics capturing different aspects of non-normality and propose their potential use in the context of the stability of complex ecosystems.**

---



PHYSICAL REVIEW E **97**, 042302 (2018)

## Topological resilience in non-normal networked systems

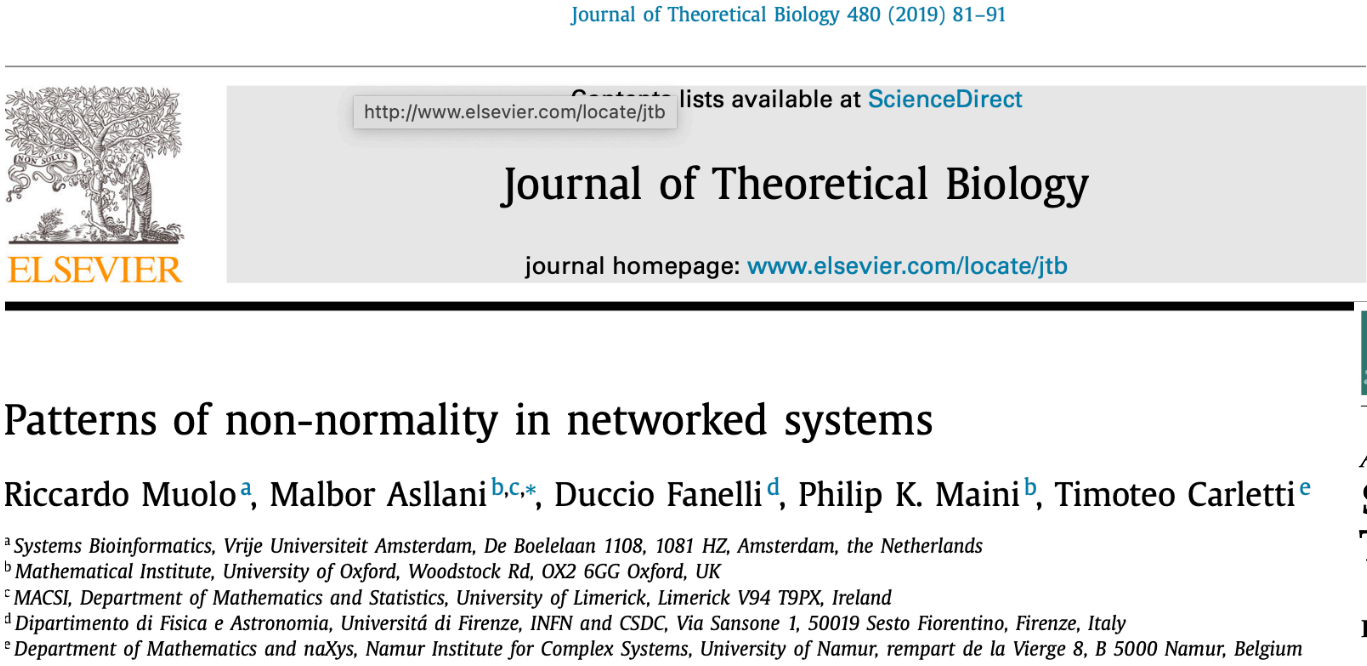
Malbor Asllani* and Timoteo Carletti
*Department of Mathematics and naXys, Namur Institute for Complex Systems,*
*University of Namur, rempart de la Vierge 8, B 5000 Namur, Belgium*

(Received 11 October 2017; revised manuscript received 19 January 2018; published 4 April 2018)

The network of interactions in complex systems strongly influences their resilience and the system capability to resist external perturbations or structural damages and to promptly recover thereafter. The phenomenon manifests itself in different domains, e.g., parasitic species invasion in ecosystems or cascade failures in human-made networks. Understanding the topological features of the networks that affect the resilience phenomenon remains a challenging goal for the design of robust complex systems. We hereby introduce the concept of non-normal networks, namely networks whose adjacency matrices are non-normal, propose a generating model, and show that such a feature can drastically change the global dynamics through an amplification of the system response to exogenous disturbances and eventually impact the system resilience. This early stage transient period can induce the formation of inhomogeneous patterns, even in systems involving a single diffusing agent, providing thus a new kind of dynamical instability complementary to the Turing one. We provide, first, an illustrative application of this result to ecology by proposing a mechanism to mute the Allee effect and, second, we propose a model of virus spreading in a population of commuters moving using a non-normal transport network, the London Tube.

DOI: 10.1103/PhysRevE.97.042302

---



Journal of Theoretical Biology 480 (2019) 81–91

http://www.elsevier.com/locate/jtb

lists available at ScienceDirect

Journal of Theoretical Biology

journal homepage: www.elsevier.com/locate/jtb

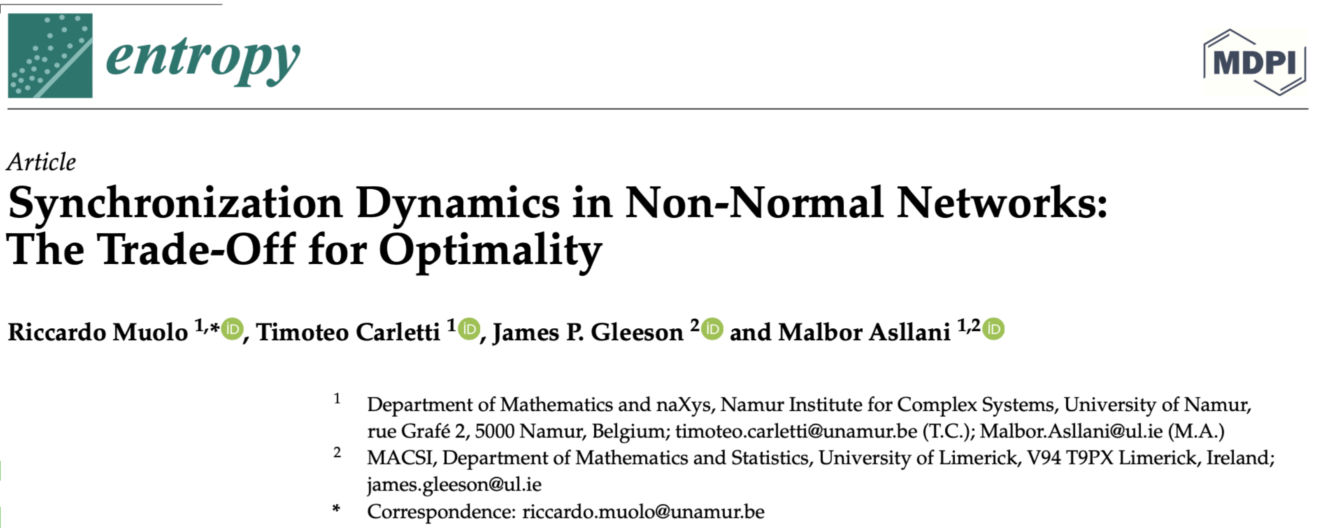
ELSEVIER

## Patterns of non-normality in networked systems

Riccardo Muolo[a], Malbor Asllani[b,c,*], Duccio Fanelli[d], Philip K. Maini[b], Timoteo Carletti[e]

[a] *Systems Bioinformatics, Vrije Universiteit Amsterdam, De Boelelaan 1108, 1081 HZ, Amsterdam, the Netherlands*
[b] *Mathematical Institute, University of Oxford, Woodstock Rd, OX2 6GG Oxford, UK*
[c] *MACSI, Department of Mathematics and Statistics, University of Limerick, Limerick V94 T9PX, Ireland*
[d] *Dipartimento di Fisica e Astronomia, Universitá di Firenze, INFN and CSDC, Via Sansone 1, 50019 Sesto Fiorentino, Firenze, Italy*
[e] *Department of Mathematics and naXys, Namur Institute for Complex Systems, University of Namur, rempart de la Vierge 8, B 5000 Namur, Belgium*

---



*entropy* MDPI

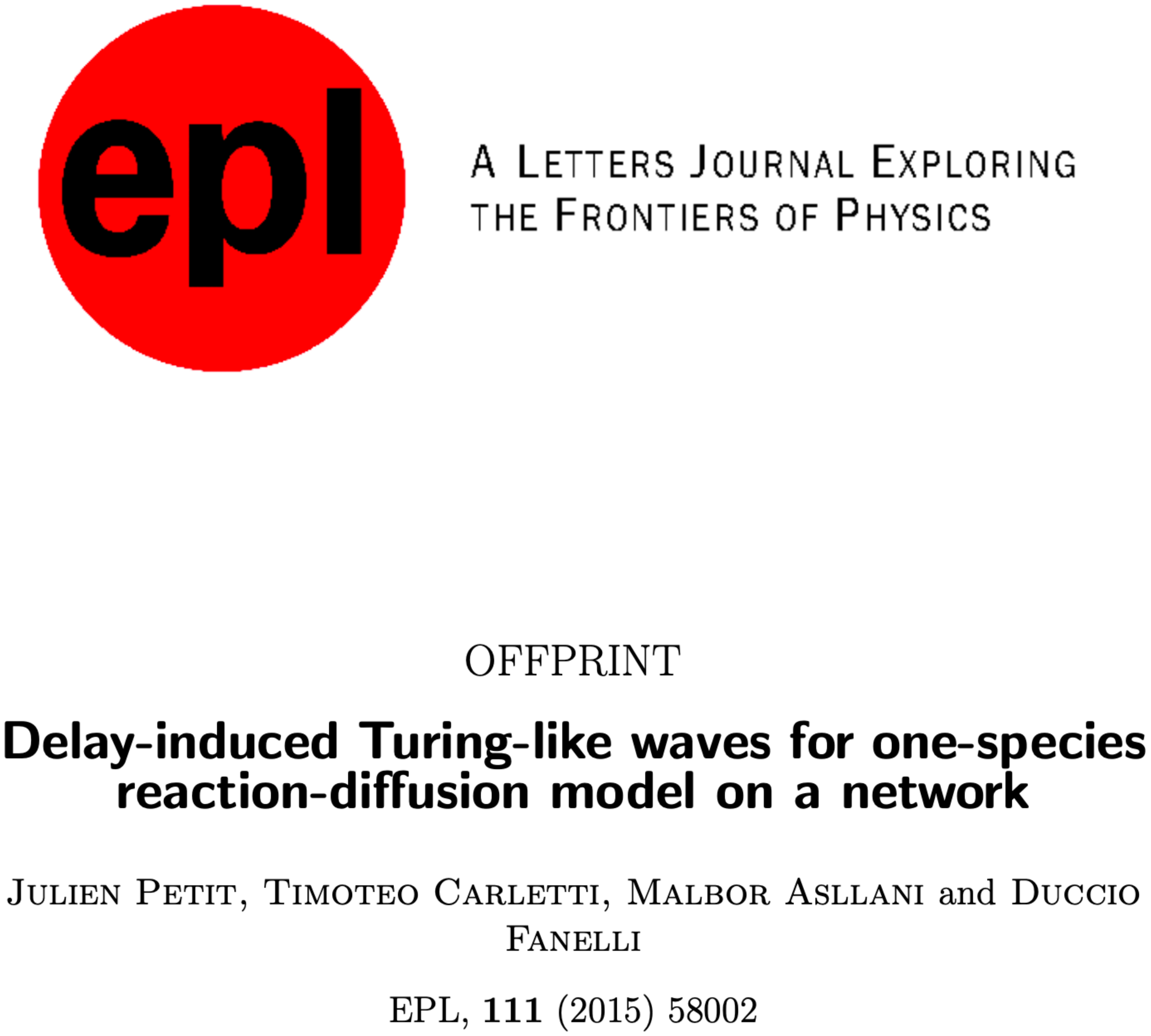
*Article*

## Synchronization Dynamics in Non-Normal Networks: The Trade-Off for Optimality

**Riccardo Muolo [1,*], Timoteo Carletti [1], James P. Gleeson [2] and Malbor Asllani [1,2]**

[1] Department of Mathematics and naXys, Namur Institute for Complex Systems, University of Namur, rue Grafé 2, 5000 Namur, Belgium; timoteo.carletti@unamur.be (T.C.); Malbor.Asllani@ul.ie (M.A.)
[2] MACSI, Department of Mathematics and Statistics, University of Limerick, V94 T9PX Limerick, Ireland; james.gleeson@ul.ie
* Correspondence: riccardo.muolo@unamur.be

# I didn’t have time but …



A Letters Journal Exploring
the Frontiers of Physics

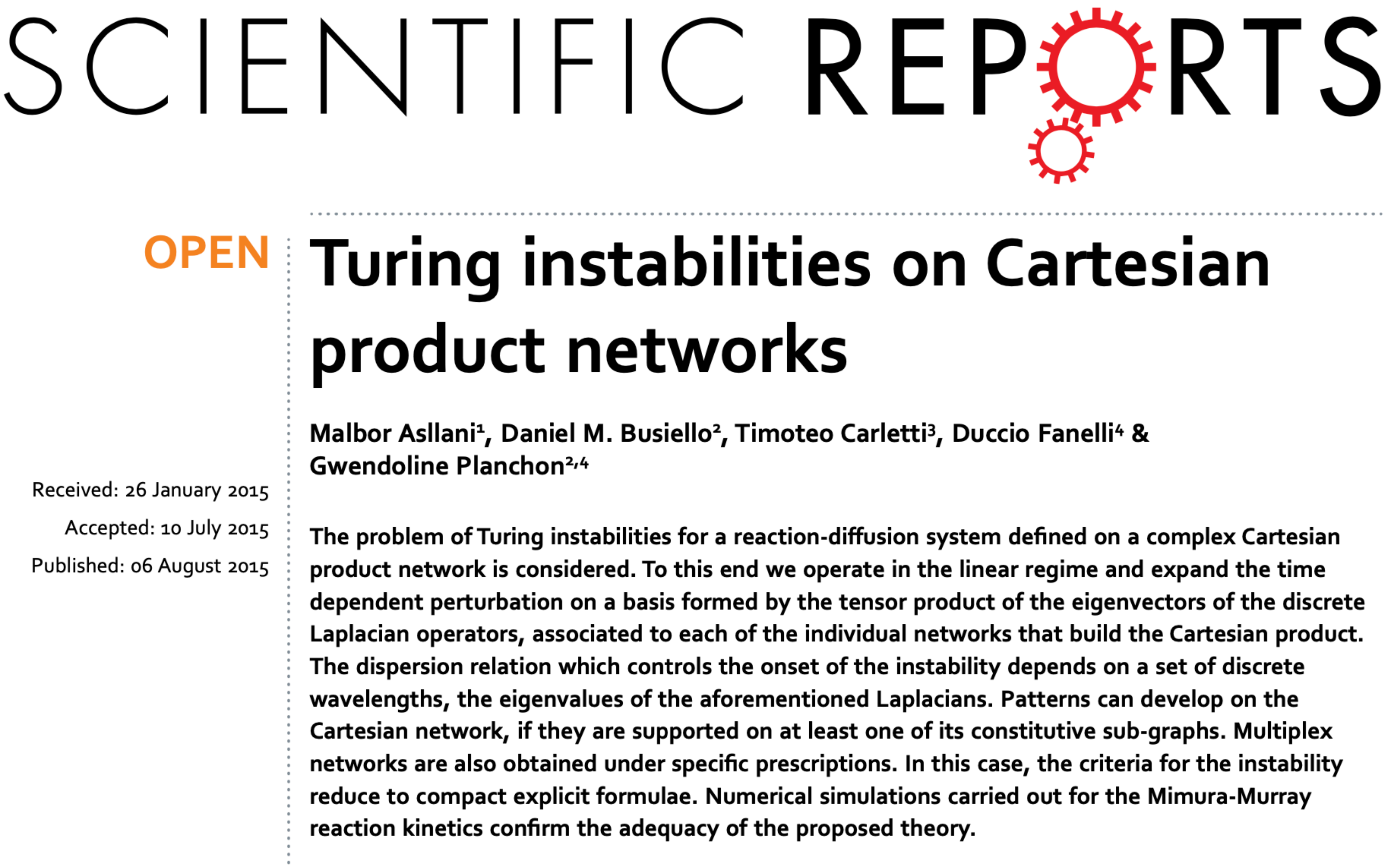
OFFPRINT

**Delay-induced Turing-like waves for one-species reaction-diffusion model on a network**

Julien Petit, Timoteo Carletti, Malbor Asllani and Duccio Fanelli

EPL, **111** (2015) 58002

# I didn't have time but …



SCIENTIFIC REPORTS

OPEN

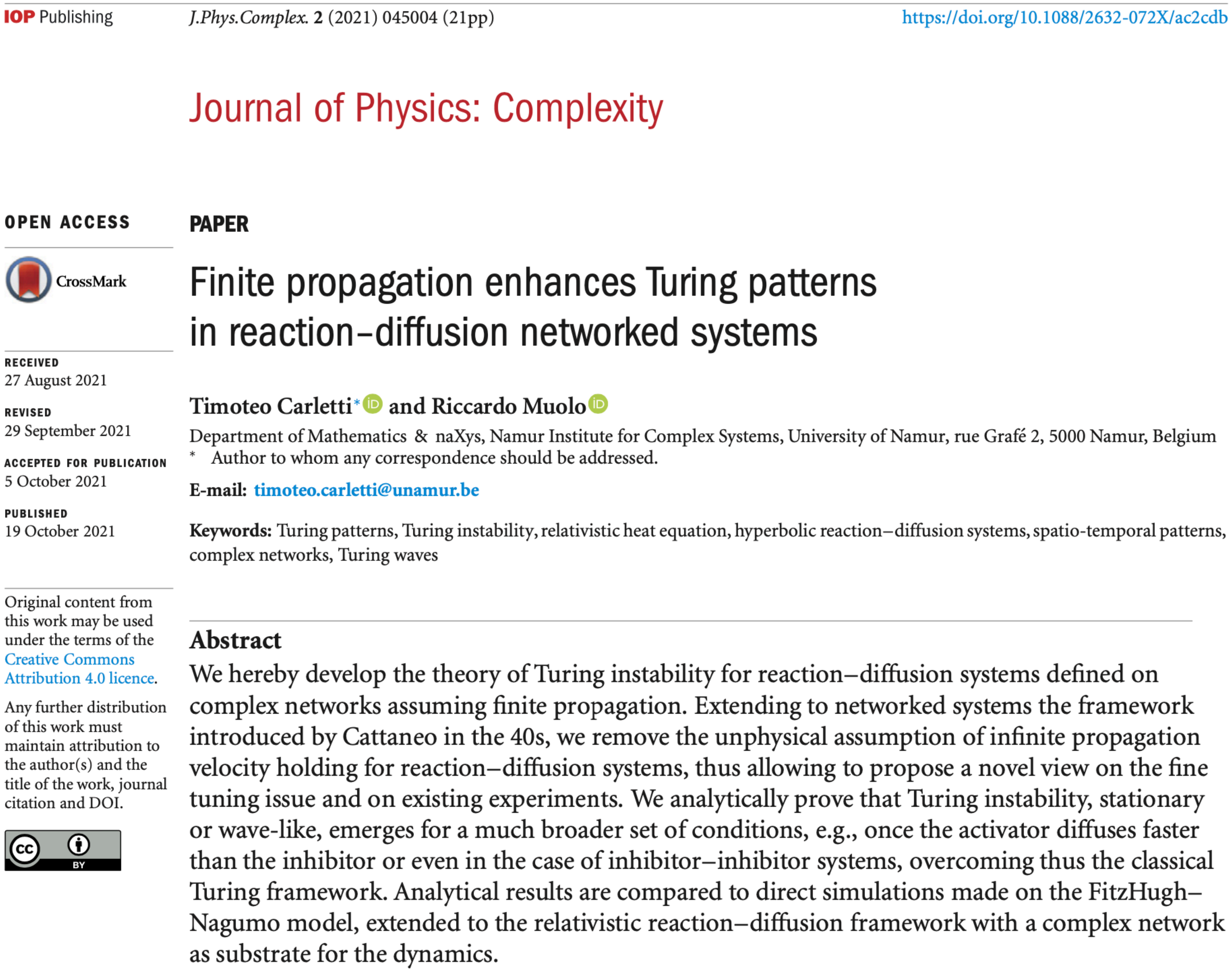
## Turing instabilities on Cartesian product networks

**Malbor Asllani[1], Daniel M. Busiello[2], Timoteo Carletti[3], Duccio Fanelli[4] & Gwendoline Planchon[2,4]**

Received: 26 January 2015
Accepted: 10 July 2015
Published: 06 August 2015

The problem of Turing instabilities for a reaction-diffusion system defined on a complex Cartesian product network is considered. To this end we operate in the linear regime and expand the time dependent perturbation on a basis formed by the tensor product of the eigenvectors of the discrete Laplacian operators, associated to each of the individual networks that build the Cartesian product. The dispersion relation which controls the onset of the instability depends on a set of discrete wavelengths, the eigenvalues of the aforementioned Laplacians. Patterns can develop on the Cartesian network, if they are supported on at least one of its constitutive sub-graphs. Multiplex networks are also obtained under specific prescriptions. In this case, the criteria for the instability reduce to compact explicit formulae. Numerical simulations carried out for the Mimura-Murray reaction kinetics confirm the adequacy of the proposed theory.

# I didn’t have time but …



IOP Publishing J.Phys.Complex. 2 (2021) 045004 (21pp) https://doi.org/10.1088/2632-072X/ac2cdb

Journal of Physics: Complexity

OPEN ACCESS

RECEIVED
27 August 2021

REVISED
29 September 2021

ACCEPTED FOR PUBLICATION
5 October 2021

PUBLISHED
19 October 2021

PAPER

## Finite propagation enhances Turing patterns in reaction–diffusion networked systems

**Timoteo Carletti*** and **Riccardo Muolo**

Department of Mathematics & naXys, Namur Institute for Complex Systems, University of Namur, rue Grafé 2, 5000 Namur, Belgium

* Author to whom any correspondence should be addressed.

**E-mail:** timoteo.carletti@unamur.be

**Keywords:** Turing patterns, Turing instability, relativistic heat equation, hyperbolic reaction–diffusion systems, spatio-temporal patterns, complex networks, Turing waves

### Abstract

We hereby develop the theory of Turing instability for reaction–diffusion systems defined on complex networks assuming finite propagation. Extending to networked systems the framework introduced by Cattaneo in the 40s, we remove the unphysical assumption of infinite propagation velocity holding for reaction–diffusion systems, thus allowing to propose a novel view on the fine tuning issue and on existing experiments. We analytically prove that Turing instability, stationary or wave-like, emerges for a much broader set of conditions, e.g., once the activator diffuses faster than the inhibitor or even in the case of inhibitor–inhibitor systems, overcoming thus the classical Turing framework. Analytical results are compared to direct simulations made on the FitzHugh–Nagumo model, extended to the relativistic reaction–diffusion framework with a complex network as substrate for the dynamics.