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Doctoral School FNRS
Nonlinear phenomena, complex systems and statistical mechanics

1-2-3 February 2022 - 9h30-12h30 – Université de Namur

Introduction to dynamical systems on complex networks

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Lecture 4 – Synchronization on complex networks

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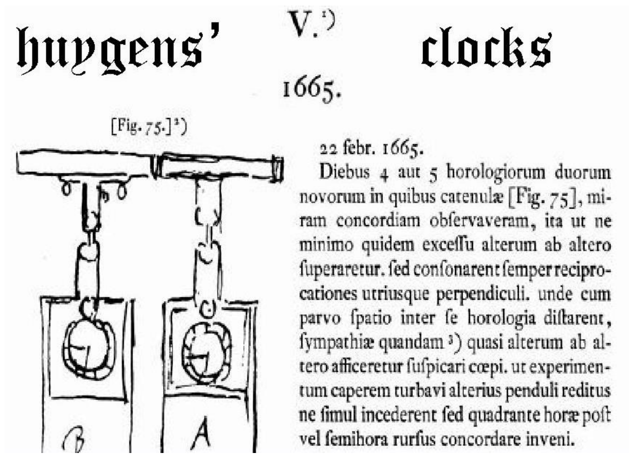
Contents

- What is synchronization?
- A simple model due to Kuramoto
- General model of coupled oscillators
- The Master Stability Function
- Control of synchronization

What is synchronization?


huygens' V.²) clocks
1665.

[Fig. 75.]¹)

22 febr. 1665.
Diebus 4 aut 5 horologiorum duorum novorum in quibus catenulæ [Fig. 75], miram concordiam obſervaveram, ita ut ne minimo quidem exceſſu alterum ab altero ſuperaretur. ſed conſonarent ſemper reciprocationes utriusque perpendiculi. unde cum parvo ſpatio inter ſe horologia diſtarent, ſympathiæ quandam ³) quasi alterum ab altero afficeretur ſuſpicari cœpi. ut experimentum caperem turbavi alterius penduli reditus ne ſimul incederent ſed quadrante horæ poſt vel ſemihora rurſus concordare inveni.


Christiaan Huygens

https://www.youtube.com/watch?v=Aaxw4zbULMs

Metronomes

Metronomes do not synchronize

Metronomes do synchronize

The model of metronomes

- A single metronome on a mobile base

$$\frac{d^2\vartheta}{dt^2}+\frac{mr_{c,m}g}{I}\sin\vartheta+\varepsilon\left[\left(\frac{\vartheta}{\vartheta_0}\right)^2-1\right]\frac{d\vartheta}{dt}+\left(\frac{mr_{c,m}\cos\vartheta}{I}\right)\frac{d^2x}{dt^2}=0$$

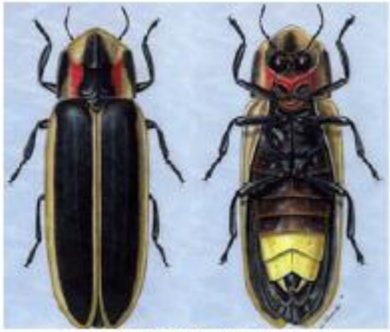
- Neglecting the damping of the base motion, the coupling term (for example for a two metronome system) can be found

$$x=-\frac{m}{M+2m}r_{c,m}\left(\sin\vartheta_1+\sin\vartheta_2\right)$$

- It can be analitically proven that metronomes synchronize [J. Pantaleone, *Am. J. Phys.*, 2002]

Systems that synchronize: a classical example

Fireflies flashing in sync on the river banks of Malaysia



Say's firefly, in the US
(Arwin Provonsha, Purdue Dept of Entomology, IN)

- Each firefly emits light flashes with a regular internal cycle
- Each firefly adjusts its lighting frequency as a function of its neighbors

Other examples

https://www.youtube.com/watch?v=eAXVa__XWZ8
https://www.youtube.com/watch?v=W-nTOo95Yy8

The Kuramoto model

- Dynamics of the oscillators (global coupling)

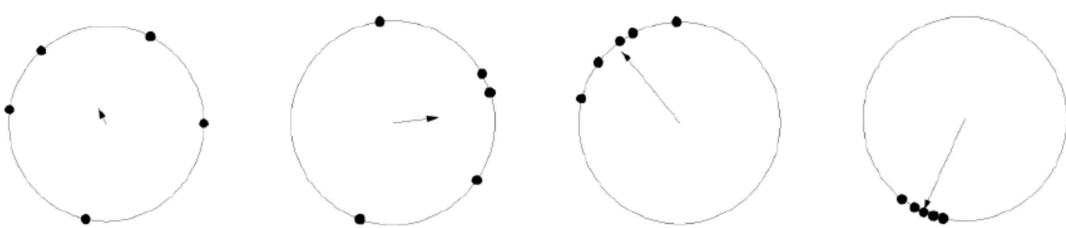
$$\dot{\vartheta}_i = \omega_i + \frac{K}{N}\sum_{j=1}^{N}\sin\left(\vartheta_j - \vartheta_i\right) \quad i = 1,\ldots,N$$

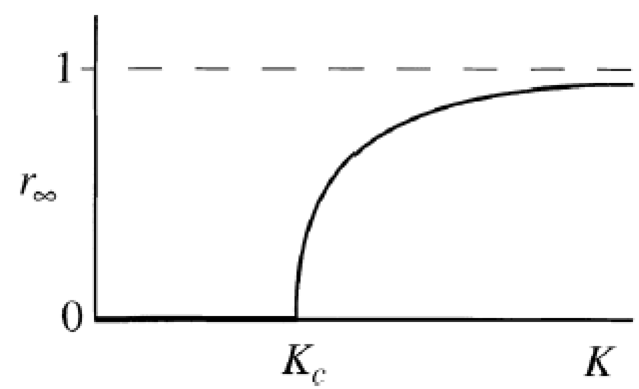
- Order parameter

$$r \cdot e^{i\psi} = \frac{1}{N}\sum_{j=1}^{N} e^{i\vartheta_j}$$

- Dynamics

$$\dot{\vartheta}_i = \omega_i + K \cdot r \cdot \sin\left(\psi - \vartheta_i\right) \quad i = 1,\ldots,N$$





Coupled oscillators

- Equations

$$\dot{x}_i = f(x_i) + \sigma \sum_{j=1}^{N} a_{ij} H(x_j - x_i) \quad i = 1,\ldots,N$$

  – f is the dynamics of each uncoupled unit (units are identical), order n
  – a_{ij} are the elements of the adjacency matrix
  – H is a constant matrix (the inner coupling)
  – σ is the coupling coefficient

- Equivalent formulation

$$\dot{x}_i = f(x_i) - \sigma \sum_{j=1}^{N} L_{ij} H x_i \quad i = 1,\ldots,N$$

Diffusive coupling

$$\dot{x}_i = f(x_i) + \sigma \sum_{j=1}^{N} a_{ij} H(x_j - x_i) \quad i = 1,\dots,N$$

$$\sum_{j=1}^{N} a_{ij} H(x_j - x_i) = \sum_{j=1}^{N} a_{ij} H x_j - \sum_{j=1}^{N} a_{ij} H x_i =$$

$$= \sum_{j=1}^{N} a_{ij} H x_j - H x_i \sum_{j=1}^{N} a_{ij} = \sum_{j=1, j\neq i}^{N} a_{ij} H x_j - d_i H x_i = -\sum_{j=1}^{N} L_{ij} H x_j$$

$$\Rightarrow \quad \dot{x}_i = f(x_i) - \sigma \sum_{j=1}^{N} L_{ij} H x_i \quad i = 1,\dots,N$$

Coupled oscillators - example

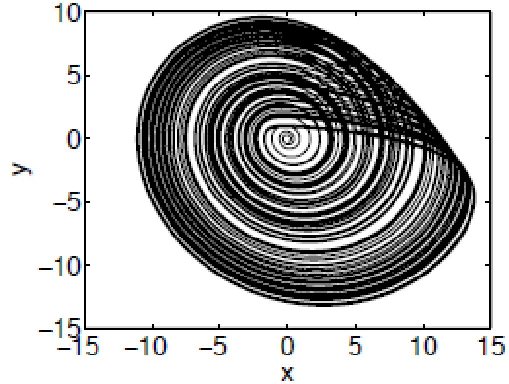
$$\dot{x}_i = f(x_i) - \sigma \sum_{j=1}^{N} L_{ij} H x_j \quad i = 1, \ldots, N$$

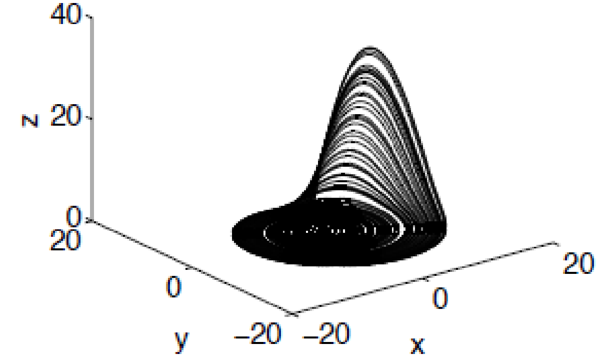
- Let us consider N Rössler units, coupled through two different configurations…

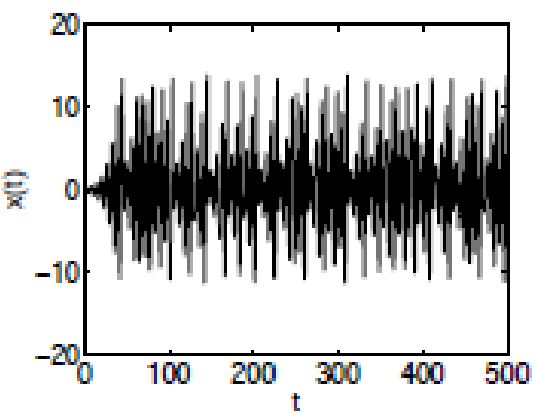
The Rössler system

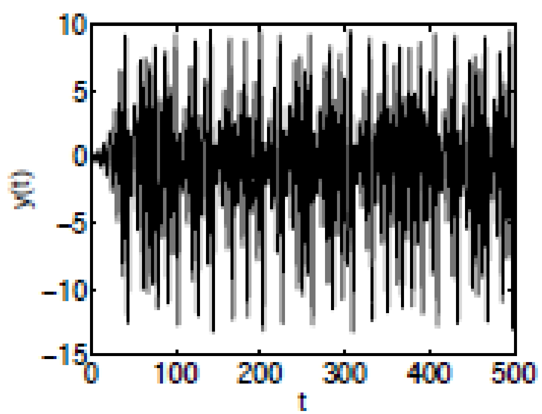
$$\dot{x} = -y - z$$
$$\dot{y} = x + ay$$
$$\dot{z} = b + z(x - c)$$

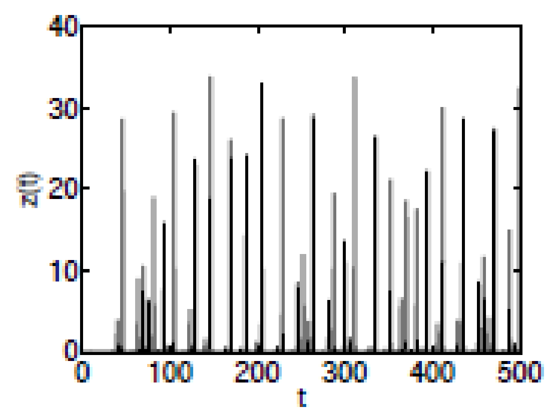
$$a = b = 0.2; c = 7$$









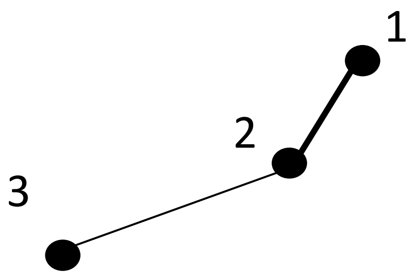


Coupled oscillators - example

$$\dot{x}_i = f(x_i) - \sigma \sum_{j=1}^{N} L_{ij} H x_j \quad i = 1, \ldots, N$$

- N=3 Rössler units, coupled through the second variable, network as in figure

$$\dot{x}_1 = -y_1 - z_1$$
$$\dot{y}_1 = x_1 + a y_1 + \sigma(y_2 - y_1)$$
$$\dot{z}_1 = b + z_1(x_1 - c)$$



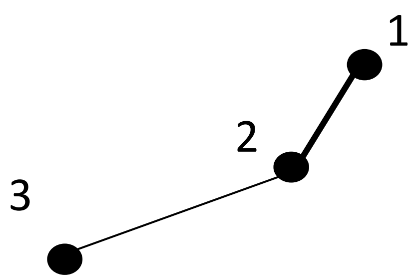
$$\dot{x}_3 = -y_3 - z_3$$
$$\dot{y}_3 = x_3 + a y_3 + \sigma(y_2 - y_3)$$
$$\dot{z}_3 = b + z_3(x_3 - c)$$

$$\dot{x}_2 = -y_2 - z_2$$
$$\dot{y}_2 = x_2 + a y_2 + \sigma(y_1 - y_2) + \sigma(y_3 - y_2)$$
$$\dot{z}_2 = b + z_2(x_2 - c)$$

Coupled oscillators - example

$$\dot{x}_i = f(x_i) - \sigma \sum_{j=1}^{N} L_{ij} H x_j \quad i = 1,\ldots,N$$

- N=3 Rössler units, coupled through the second variable, network as in figure



$$H = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$H\boldsymbol{x}_j = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} x_j \\ y_j \\ z_j \end{pmatrix} = \begin{pmatrix} 0 \\ y_j \\ 0 \end{pmatrix}$$

$$L = \begin{pmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{pmatrix}$$

$$\begin{aligned} \dot{x}_1 &= -y_1 - z_1 \\ \dot{y}_1 &= x_1 + a y_1 \boxed{+ \sigma(y_2 - y_1)} \\ \dot{z}_1 &= b + z_1(x_1 - c) \end{aligned}$$

The synchronization error

- Synchronization error

$$e(t) = \left(\frac{1}{N(N-1)} \sum_{i,j} \| x_i(t) - x_j(t) \|^2 \right)^{\frac{1}{2}}$$

- The network is (globally) synchronized if

$$\lim_{t \to +\infty} e(t) = 0$$

- Equivalent definition of the error

$$e(t) = \left(\frac{2}{N-1} \sum_{i} \| x_i(t) - \bar{x}(t) \|^2 \right)^{\frac{1}{2}}$$

with

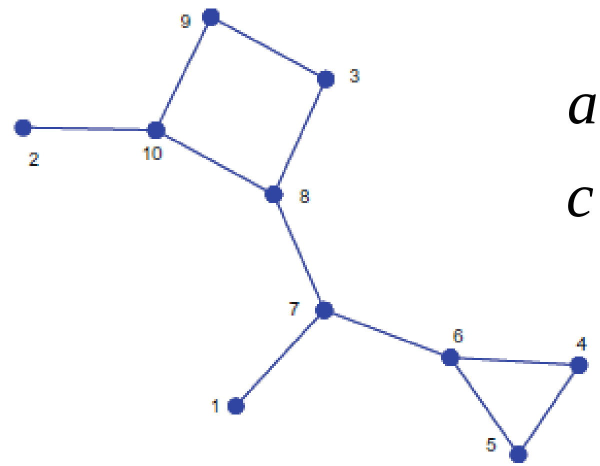
$$\bar{x}(t) = \frac{1}{N} \sum_i x_i(t)$$

A network of Rössler oscillators

$$\dot{x}_i = -y_i - z_i$$

$$\dot{y}_i = x_i + a y_i - \sigma \sum_{j=1}^{N} L_{ij} y_j$$

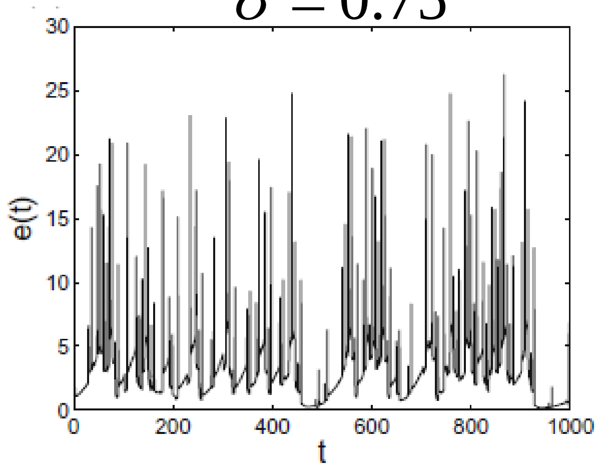
$$\dot{z}_i = b + z_i (x_i - c)$$



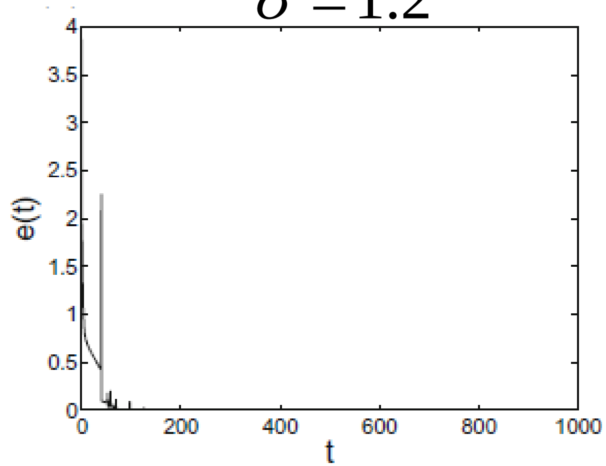
$a = b = 0.2$

$c = 9$

$\sigma = 0.75$



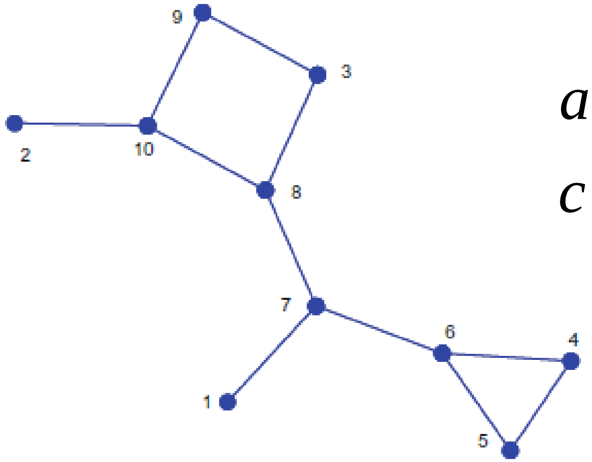
$\sigma = 1.2$



The network, which was not synchronized for small coupling, becomes synchronized for larger coupling

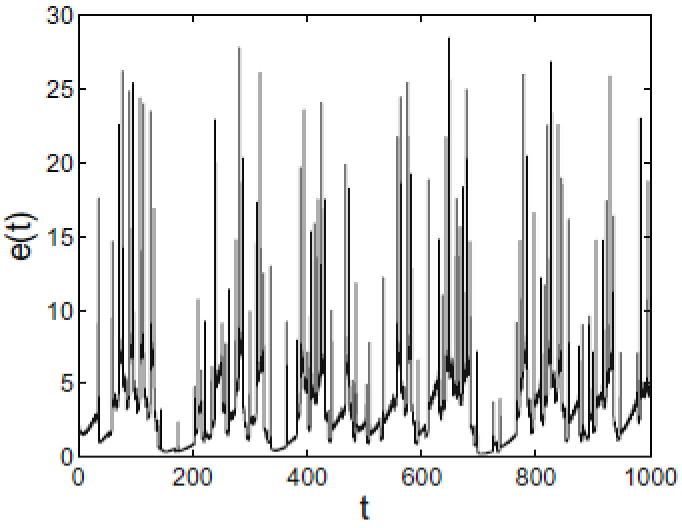
Same network, different coupling

$$\dot{x}_i = -y_i - z_i - \sigma \sum_{j=1}^{N} L_{ij} x_j$$
$$\dot{y}_i = x_i + a y_i$$
$$\dot{z}_i = b + z_i (x_i - c)$$



$$a = b = 0.2$$
$$c = 9$$

$$\sigma = 0.75$$



Synchronization is never achieved, even if the coupling strength is increased!

Summing up…

- Two networks: one synchronizes for large enough coupling strength, the other does not (for any coupling strength)

- However, the coupling is always diffusive, and since

$$L\vec{1} = 0$$

the synchronous manifold $x_1=x_2=\ldots=x_N=x_s$ exists in both cases!

$$\dot{x}_s = f(x_s)$$

- What is the difference here?

Stability of the synchronous state

- Reference model

$$\dot{x}_i = f(x_i) - \sigma \sum_{j=1}^{N} L_{ij} h(x_j) \quad i = 1, \ldots, N$$

- Synchronous state

$$x_1(t) = \ldots = x_N(t) = s(t)$$

with

$$\dot{s} = f(s)$$

- Compact form

$$\dot{\mathbf{x}} = \mathbf{F}(\mathbf{x}) - \sigma L \otimes I_n \mathbf{H}(\mathbf{x})$$

where

$$\mathbf{x} = \begin{bmatrix} x_1 \\ \vdots \\ x_N \end{bmatrix} \qquad \mathbf{F}(\mathbf{x}) = \begin{bmatrix} f(x_1) \\ \vdots \\ f(x_N) \end{bmatrix} \qquad \mathbf{H}(\mathbf{x}) = \begin{bmatrix} h(x_1) \\ \vdots \\ h(x_N) \end{bmatrix}$$

Stability of the synchronous state

- Let's define the new variables

$$\eta_i(t) = x_i(t) - s(t)$$

- Linearization around s(t)

$$\dot{\boldsymbol{\eta}} = (I_N \otimes Df)\boldsymbol{\eta} - \sigma(L \otimes Dh)\boldsymbol{\eta}$$

with

$$Df = \left.\frac{\partial f(x_i)}{\partial x_i}\right|_{x_i(t)=s(t)} \qquad Dh = \left.\frac{\partial h(x_i)}{\partial x_i}\right|_{x_i(t)=s(t)}$$

- Consider now

$$\boldsymbol{\xi} = (T^{-1} \otimes I_n)\boldsymbol{\eta}$$

with T such that

$$T^{-1}LT = diag(\lambda_1, ..., \lambda_N)$$

Stability of the synchronous state

$$\dot{\boldsymbol{\xi}} = (T^{-1} \otimes I_n)(I_N \otimes Df)(T \otimes I_n)\boldsymbol{\xi} - \sigma(T^{-1} \otimes I_n)(L \otimes Dh)(T \otimes I_n)\boldsymbol{\xi}$$

$$\Downarrow$$

$$\dot{\boldsymbol{\xi}} = (I_N \otimes Df)\boldsymbol{\xi} - \sigma(T^{-1}LT \otimes Dh)\boldsymbol{\xi}$$

- Blocks are identical

$$\dot{\xi}_h = (Df - \sigma\lambda_h Dh)\xi_h$$

- Master stability equation

$$\dot{\zeta} = (Df - \alpha Dh)\zeta$$

- Master stability function (MSF)

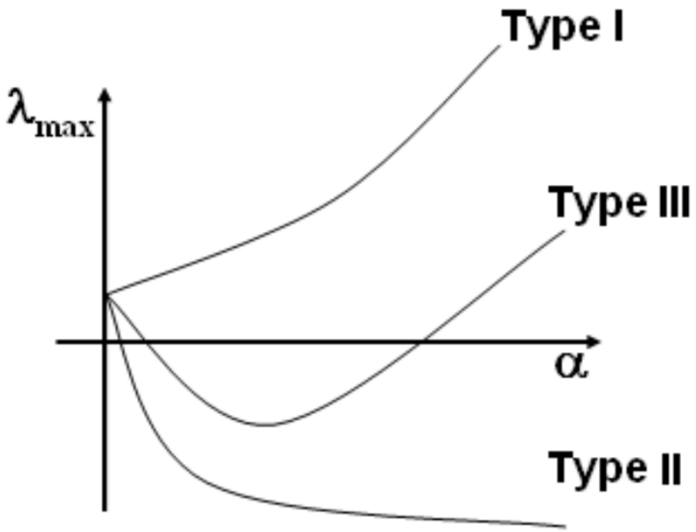
$$\lambda_{\max} = \lambda_{\max}(\alpha)$$

The Master Stability Function

[Pecora and Carroll, 1998]

$$\dot{\varsigma} = [DF + (\alpha + i\beta)DH]\varsigma$$

Considering real eigenvalues (i.e. β=0)



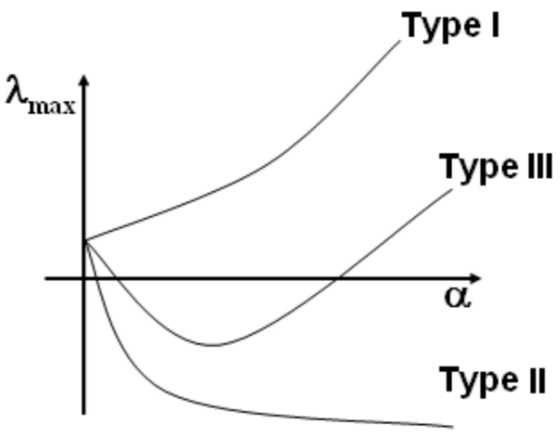
Synchronization

$$\lambda_{\max}(\sigma\lambda_h) < 0$$

$\forall \lambda_h$ eigenvalue of L

The Master Stability Function

[Pecora and Carroll, 1998]



- Type I: networks never synchronizable
- Type II:

$$\sigma > \alpha_1 / \lambda_2$$

- Type III:

$$\sigma > \alpha_1 / \lambda_2$$

$$\sigma < \alpha_1 / \lambda_N$$

Requires that:

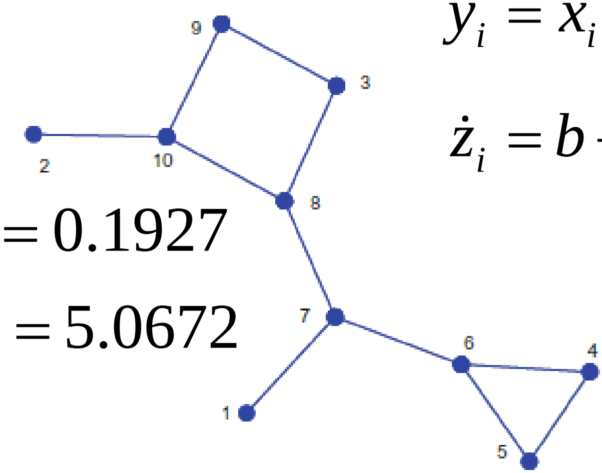
$$\frac{\lambda_N}{\lambda_2} < \frac{\alpha_2}{\alpha_1}$$

Network of Rössler oscillators

$$\dot{x}_i = -y_i - z_i$$
$$\dot{y}_i = x_i + a y_i - \sigma \sum_{j=1}^{N} L_{ij} y_j$$
$$\dot{z}_i = b + z_i (x_i - c)$$

Type II

$$\alpha_1 = 0.157$$



$$\lambda_2 = 0.1927$$
$$\lambda_N = 5.0672$$

Type III

$$\dot{x}_i = -y_i - z_i - \sigma \sum_{j=1}^{N} L_{ij} x_j$$
$$\dot{y}_i = x_i + a y_i$$
$$\dot{z}_i = b + z_i (x_i - c)$$

$$\alpha_1 = 0.186$$
$$\alpha_2 = 4.614$$

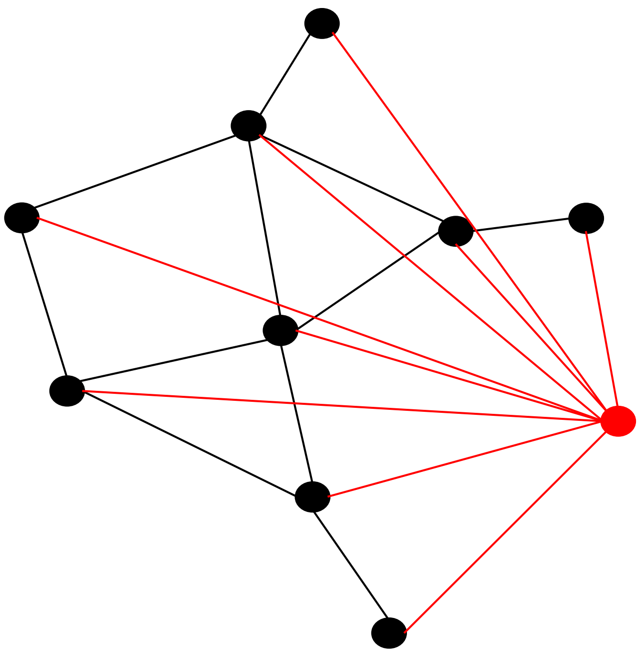
$$26.3 = \frac{\lambda_N}{\lambda_2} \cancel{<} \frac{\alpha_2}{\alpha_1} = 24.8$$

Control of synchronization

- A network of coupled chaotic oscillators may not satisfy the conditions for synchronization (e.g., the eigenvalues of the Laplacian are outside the bounds of master stability function, MSF)

- Even for networks reaching synchronization, control may be needed to steer the system towards a specific trajectory (opposed to a self-organized solution)

How can we control a network?



Coupled dynamics

$$\dot{x}_i = f(x_i) + \sigma \sum_j a_{ij} h(x_j, x_i)$$

Coupled dynamics with control

$$\dot{x}_i = f(x_i) + \sigma \sum_j a_{ij} h(x_j, x_i) \boxed{+ u_i}$$

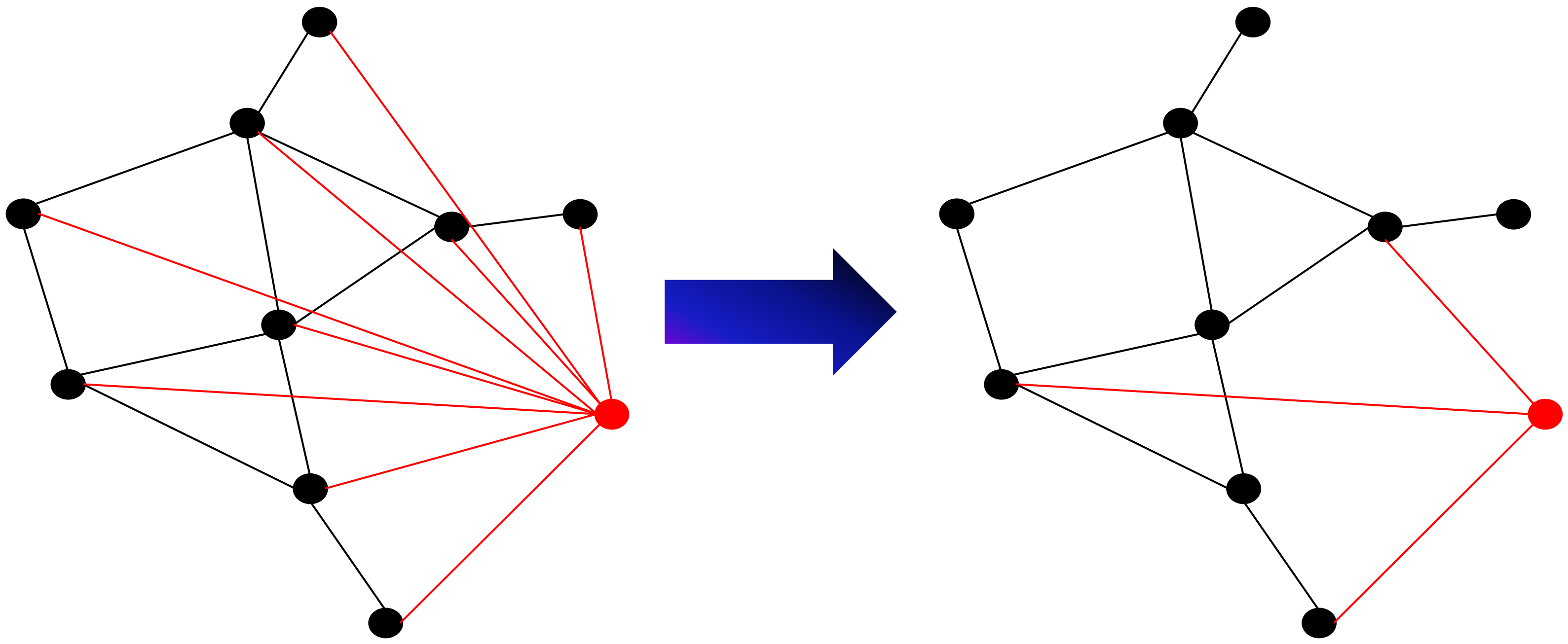
control

$$u_i = k(x_{ref} - x_i)$$

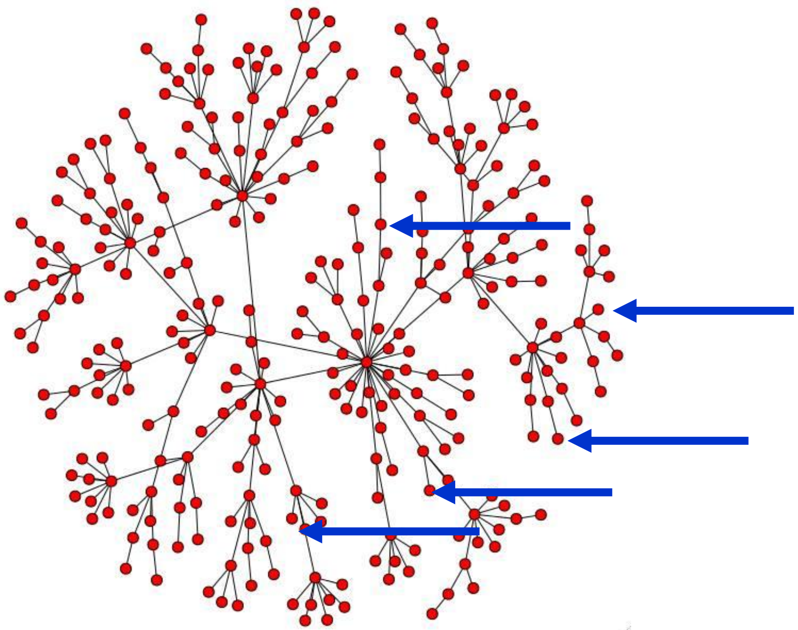
$$\dot{x}_{ref} = g(x_{ref})$$

Can we do better?

- Can we control the whole network by applying feedback only to a subset of nodes?



Pinning control



- It is a feedback control strategy for synchronization and consensus in complex dynamical networks
- A virtual leader (the *pinner*) is added to the network and defines the desired trajectory, controlling only a small fraction of the network nodes (the *pinned nodes*)
- The control action is a function of the pinning error vector, whose *i*-th component is given by the difference between the output of the pinner and the output of the *i*-the node
- Classical pinning control targets a homogeneous collective behavior

Pinning control of synchronization

Network units $\dot{x}_i = f(x_i) + \sigma \sum_j a_{ij}(x_j - x_i) + u_i;\ i = 1, \dots N$

Virtual node $\dot{s} = f(s)$

Control $u_i = k\delta_i(s - x_i)$

Pinned nodes $\delta_i = 1$

Augmented network, $x_{N+1}(t)=s(t)$

$$\dot{x}_i = f(x_i) + \sum_j w_{ij}\,(x_j - x_i);\ \mathrm{i} = 1, \dots, \mathrm{N} + 1$$

Pinning control of synchronization

$$\dot{x}_i = f(x_i) + \sum_j w_{ij}\,(x_j - x_i);\ \mathrm{i} = 1, \dots, \mathrm{N} + 1$$

Control is achieved iff the augmented network synchronizes

$$\lim_{t \to +\infty} \left\| x_i - x_j \right\| = 0 \quad \longleftrightarrow \quad \lim_{t \to +\infty} \left\| x_i - s \right\| = 0$$

F. Sorrentino, M. di Bernardo, F. Garofalo, and G. Chen, Controllability of complex networks via pinning, *Phys. Rev. E* 75, 046103, 2007

Pinning control of synchronization

How to select the nodes to pin? Is there a relationship with some topological properties?

- [Wang & Chen, Phys. A, 2002]: degree-based selective approaches are a preferable strategy with respect to random choices of nodes.

- [Chen, Liu, Lu, TCAS I, 2007]: if the directed network admits a spanning tree, then control can be achieved by only pinning the root node.

- [Lu, Li, Rong, Automatica, 2010]: if the network does not admit a spanning tree, then at least one node in each root strongly connected component has to be pinned.

Control of synchronization in a group of nodes

L. V. Gambuzza, M. Frasca, V. Latora, "Distributed control of synchronization of a group of network nodes", IEEE Trans. Automatic Control, March 2019.

Problem statement

Consider the dynamical network formed by N diffusively coupled identical nodes

$$\dot{\mathbf{x}}_i = f(\mathbf{x}_i) - \sigma \sum_{j=1}^{N} \mathcal{L}_{ij} \mathrm{H} \mathbf{x}_j + \mathbf{u}_i$$

where x_i are the state variable of node *i*, $\sigma>0$ is the coupling strength and H is a nxn constant matrix of 0-1 coefficients, find the network distributed controllers $\mathbf{u}_i$ of the form

$$\mathbf{u}_i = -\sigma \sum_{j=1}^{N} \mathcal{L'}_{ij} \mathrm{H} \mathbf{x}_j$$

such that the nodes of an arbitrary set, say S_{n2}, synchronize to each other

Main result

Assumption 1. Consider the dynamical network (10). Suppose that there exist a diagonal matrix $\mathrm{C} > 0$ and two constants $\bar{d} > 0$ and $\tau > 0$, such that

$$[Df(\mathbf{s}(t)) - d\mathrm{H}]^T \mathrm{C} + \mathrm{C}[Df(\mathbf{s}(t)) - d\mathrm{H}] \leq -\tau \mathrm{I}_n$$

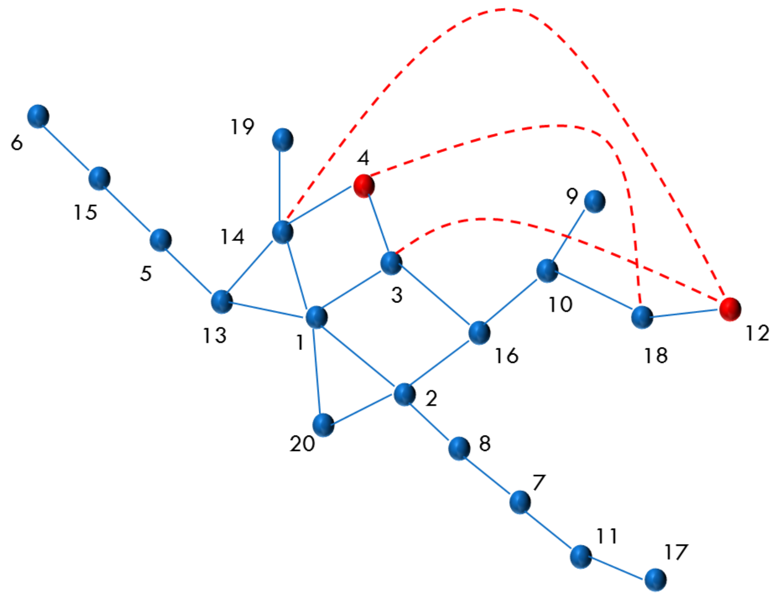
for all $d \geq \bar{d}$.

Theorem 4.1: Consider the dynamical network (10) satisfying Assumption 1, a set $\mathcal{S}_{n_2}$ of n_2 arbitrary nodes, and a fixed value of the coupling coefficient such that $\sigma > \bar{d}/n_1$, then the synchronous state $\mathbf{x}_s = [\mathbf{s}_1(t)^T \ \ldots \ \mathbf{s}_{n_1}(t)^T \ \mathbf{s}(t)^T \ \ldots \ \mathbf{s}(t)^T]^T$ is stabilized by the controllers $\mathbf{u}_i$

$$\mathbf{u}_i = -\sigma \sum_{j=1}^{N} \mathcal{L}'_{ij} \mathrm{H} \mathbf{x}_j \qquad (14)$$

with $\mathcal{L}'$ such that $\mathcal{L}'' = \mathcal{L} + \mathcal{L}'$ satisfy these conditions:

1) $\mathcal{N}''_{n_1+1} = \ldots = \mathcal{N}''_N$; from this condition it follows that $k_{n_1+1} = \ldots = k_N$, that is, the nodes of $\mathcal{S}_{n_2}$ have the same degree, we indicate it as κ, i.e., $k_{n_1+1} = \ldots = k_N \triangleq \kappa$;
2) κ is selected such that $\kappa > \frac{\bar{d}}{\sigma}$;
3) the n_2 nodes are not connected each other.



For two nodes, A and B

if $|\mathcal{N}_A \cup \mathcal{N}_B| > \bar{d}/\sigma$

$$\mathbf{u}_A = \sigma \sum_{j \in (\mathcal{N}_B - \mathcal{N}_A)} \mathrm{H}(\mathbf{x}_j - \mathbf{x}_A)$$

$$\mathbf{u}_B = \sigma \sum_{j \in (\mathcal{N}_A - \mathcal{N}_B)} \mathrm{H}(\mathbf{x}_j - \mathbf{x}_B)$$

---

if $|\mathcal{N}_A \cup \mathcal{N}_B| < \bar{d}/\sigma$

$$\mathcal{V}_2 \subseteq (\mathcal{V}(\mathcal{G}) - \mathcal{N}_A - \mathcal{N}_B - \{A, B\})$$

$$|\mathcal{N}_A \cup \mathcal{N}_B| + |\mathcal{V}_2| > \bar{d}/\sigma$$

$$\mathbf{u}_A = \sigma \sum_{j \in (\mathcal{N}_B - \mathcal{N}_A)} \mathrm{H}(\mathbf{x}_j - \mathbf{x}_A) + \sigma \sum_{j \in \mathcal{V}_2} \mathrm{H}(\mathbf{x}_j - \mathbf{x}_A)$$

$$\mathbf{u}_B = \sigma \sum_{j \in (\mathcal{N}_A - \mathcal{N}_B)} \mathrm{H}(\mathbf{x}_j - \mathbf{x}_B) + \sigma \sum_{j \in \mathcal{V}_2} \mathrm{H}(\mathbf{x}_j - \mathbf{x}_B)$$

Sketch of the proof

Dynamical network with control

$$\dot{\mathbf{x}} = F(\mathbf{x}) - \sigma \mathcal{L}'' \otimes \mathrm{H} \cdot \mathbf{x}$$

1. Linearization

$$\dot{\eta} \quad = \quad \mathrm{DF} \cdot \eta - (\mathcal{L}'' \otimes \mathrm{H}) \cdot \eta$$

2. State transformation

$$\mathrm{R} = \begin{bmatrix} \mathrm{I}_{n_1} & 0 \\ 0 & \mathbf{1} \cdot \mathbf{1}^T - \mathrm{I}_{n_2} \end{bmatrix}$$

$$\mathrm{M}^{-1}\mathrm{RM} \quad = \quad \mathrm{diag}\{\lambda_1(\mathrm{R}), \ldots, \lambda_N(\mathrm{R})\}$$

Sketch of the proof

Dynamical network with control

$$
\dot{\xi} = \begin{bmatrix} Df_1 & & & & & & & & \\ & Df_2 & & & & & & & \\ & & \ddots & & & & & & \\ & & & \ddots & & & & & \\ & & & & Df_{n_1} & & & & \\ & & & & & Df_s & & & \\ & & & & & & Df_s & & \\ & & & & & & & \ddots & \\ & & & & & & & & Df_s \end{bmatrix} \cdot \xi - \sigma \left[\begin{array}{ccccc|cccc} & & & & & 0 & 0 & \dots & 0 \\ & & & & & \vdots & & & \\ & & \mathcal{L}''_{11} & & & \frac{N}{\sqrt{N}} & 0 & \dots & 0 \\ & & & & & \vdots & & & \\ & & & & & 0 & 0 & \dots & 0 \\ \hline 0 & \dots & \frac{N}{\sqrt{N}} & \dots & 0 & \kappa & & & \\ 0 & \dots & 0 & \dots & 0 & & \kappa & & \\ \vdots & & & & & & & \ddots & \\ 0 & \dots & 0 & \dots & 0 & & & & \kappa \end{array} \right] \otimes \mathrm{H} \cdot \xi
$$

Dynamics of the transverse modes (the blocks are identical!)

$$
\dot{\zeta} = (Df - \sigma\kappa\mathrm{H})\zeta
$$

Remarks

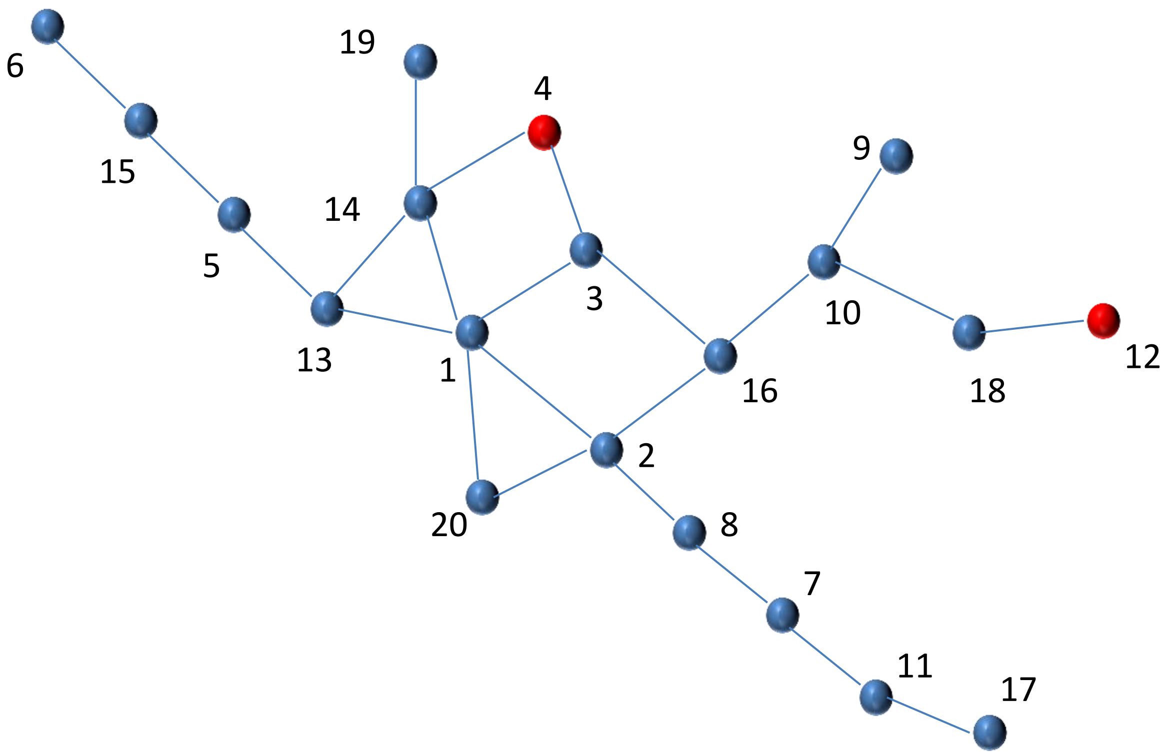
- It is not important to which nodes the nodes to synchronize are connected but only their number

- The approach can also apply within the MSF framework

- The results can be extended to the case of heterogeneous units, provided that the nodes to be synchronized have the same dynamics

An example

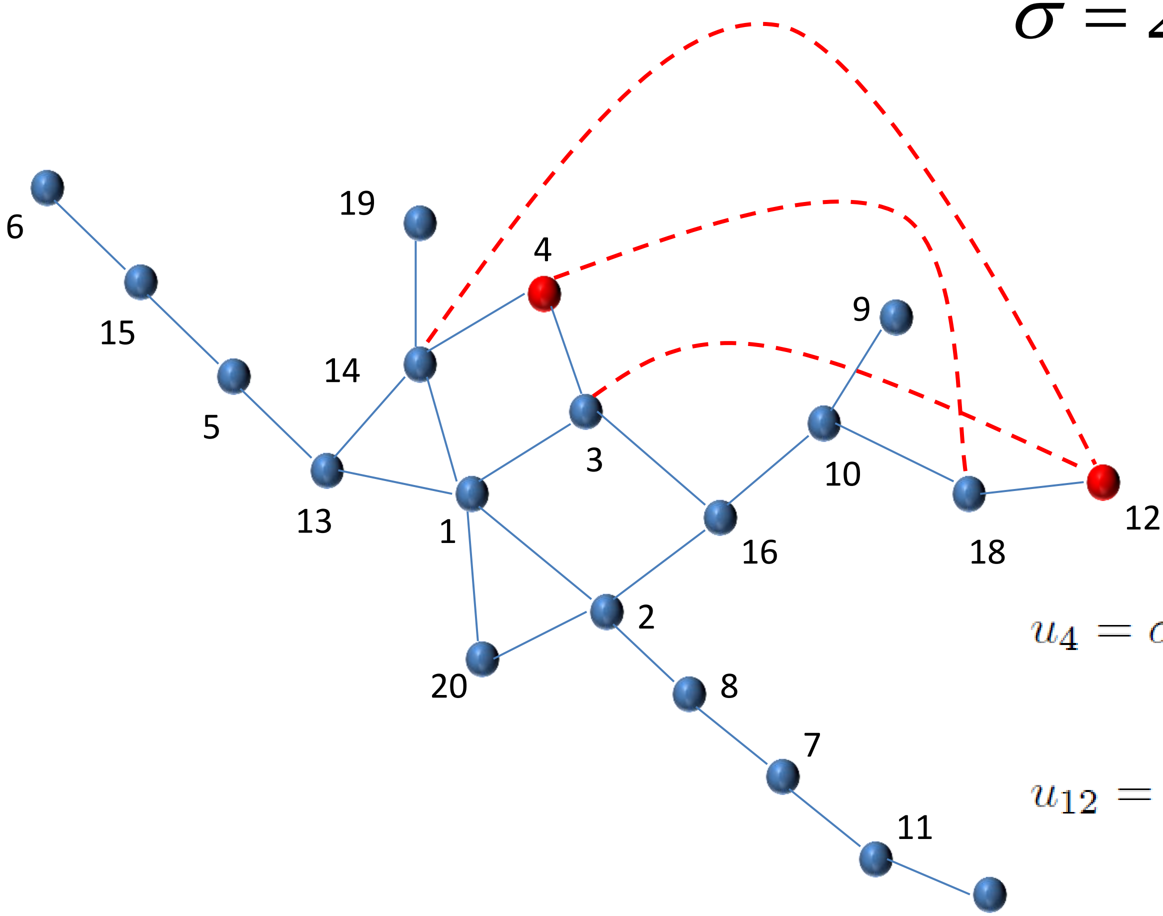
- Chua's oscillators
- Nodes to synchronize: 4 and 12



An example

$$\sigma = 2, \quad \bar{\mathrm{d}} = 4.5, \quad \kappa > \frac{\bar{d}}{\sigma}$$

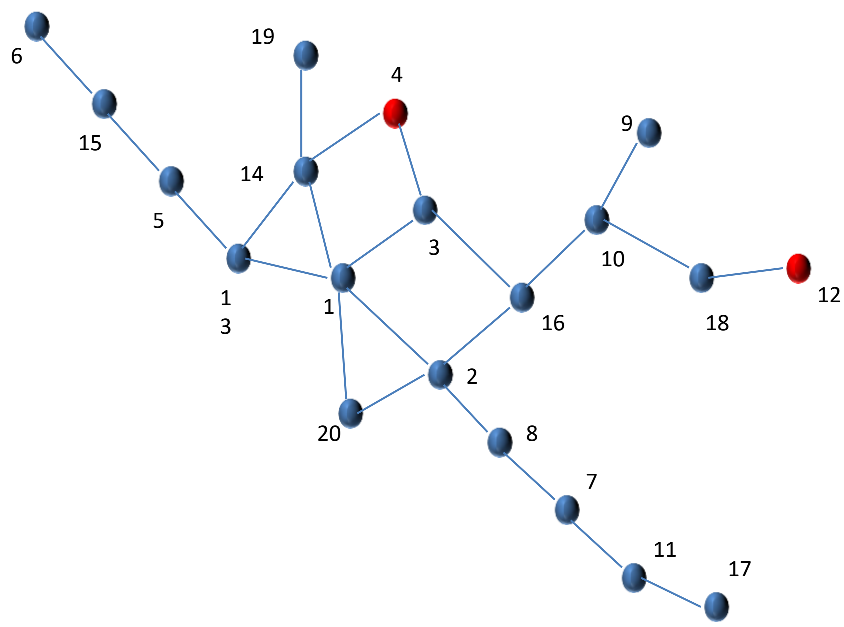
$$\kappa = 3$$



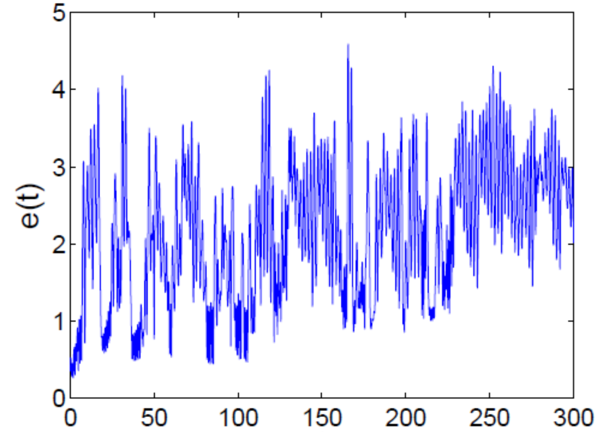
$$u_4 = \sigma \sum_{j \in (\mathcal{N}_{12} - \mathcal{N}_4) = \{18\}} \mathrm{H}(\mathbf{x}_j - \mathbf{x}_4)$$

$$u_{12} = \sigma \sum_{j \in (\mathcal{N}_4 - \mathcal{N}_{12}) = \{3,14\}} \mathrm{H}(\mathbf{x}_j - \mathbf{x}_{12})$$

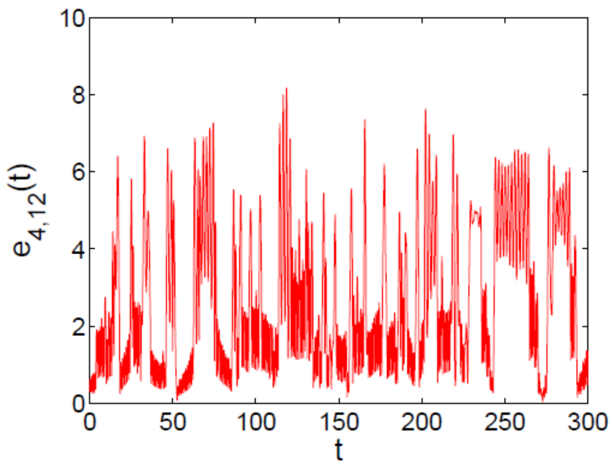
Without control



$$e(t) = \sqrt{\frac{1}{N^2}\sum_{i,j} \| \mathbf{x}_i(t) - \mathbf{x}_j(t) \|^2}$$

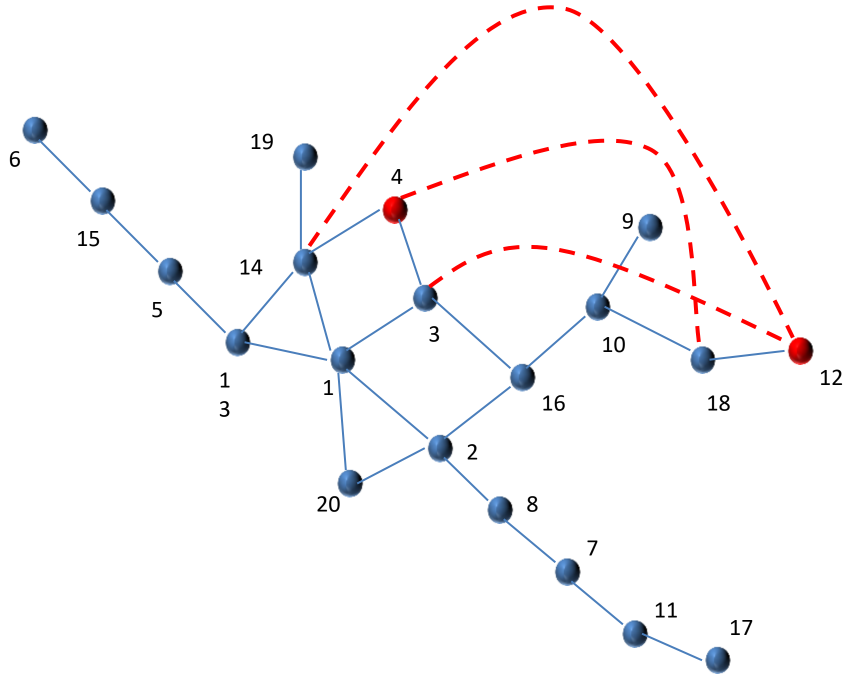


$$e_{4,12}(t) = \| \mathbf{x}_4(t) - \mathbf{x}_{12}(t) \|$$

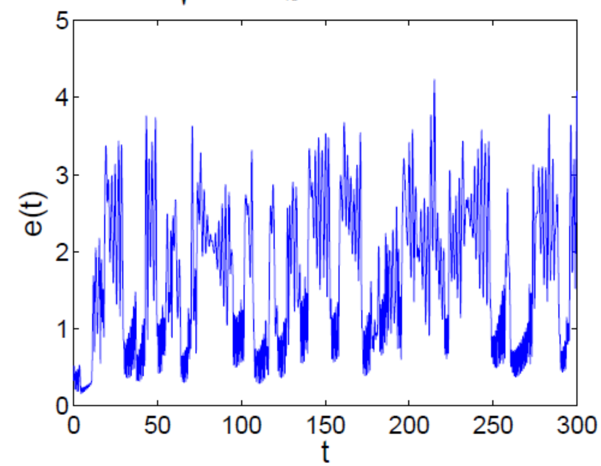


For the given value of σ, the network without control is not synchronized

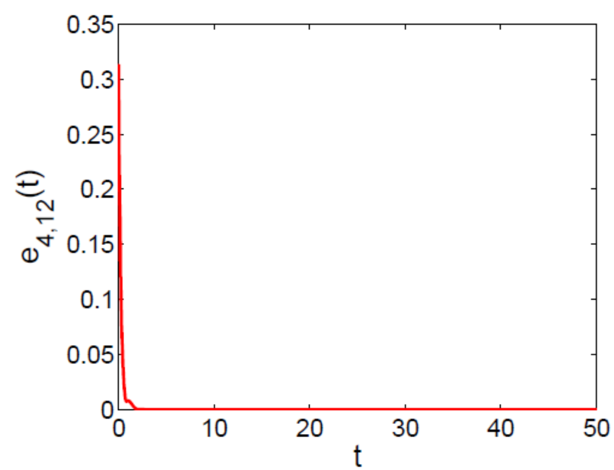
With control



$$e(t) = \sqrt{\frac{1}{N^2}\sum_{i,j} \| \mathbf{x}_i(t) - \mathbf{x}_j(t) \|^2}$$

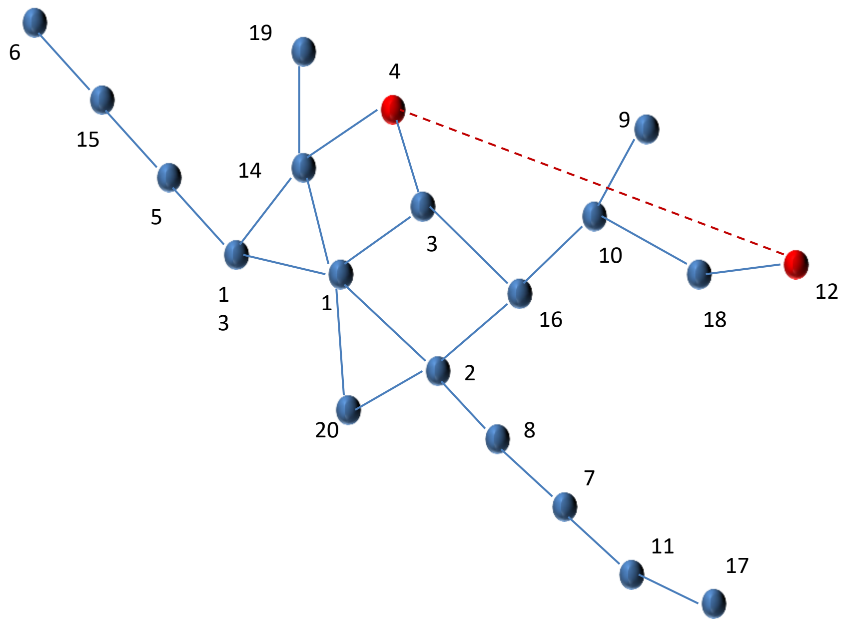


$$e_{4,12}(t) = \| \mathbf{x}_4(t) - \mathbf{x}_{12}(t) \|$$

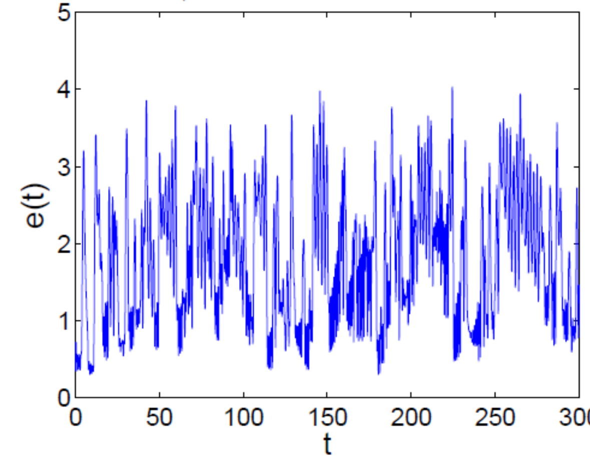


Control enables synchronization between node 4 and 12, while the rest of the network remains not synchronized

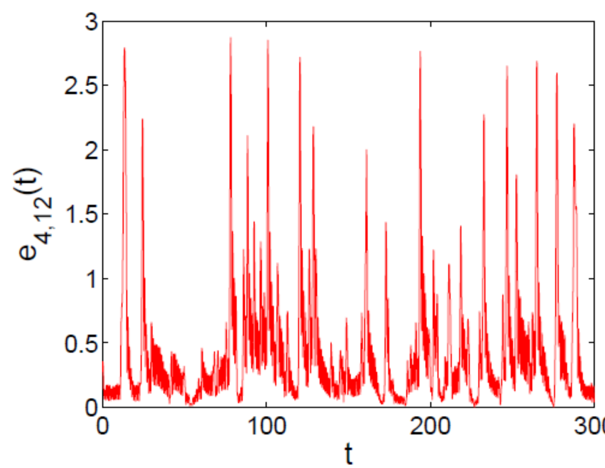
With a direct link between nodes 4 and 12



$$e(t) = \sqrt{\frac{1}{N^2}\sum_{i,j} \parallel \mathbf{x}_i(t) - \mathbf{x}_j(t) \parallel^2}$$



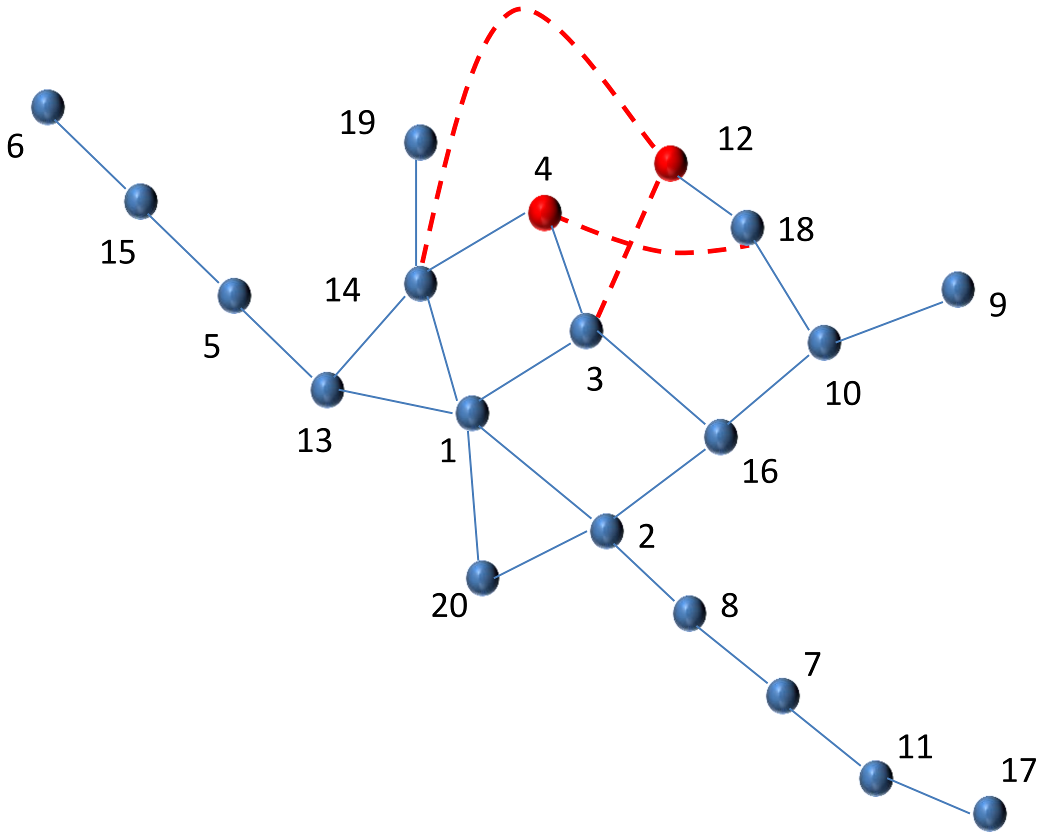
$$e_{4,12}(t) = \parallel \mathbf{x}_4(t) - \mathbf{x}_{12}(t) \parallel$$



The introduction of a direct link between nodes 4 and 12 is not sufficient to induce synchronization in them

Take home message

- Act on their neighbors!



Summary

- Examples of synchronization, models
- Synchronization of dynamical oscillators on complex networks: existence and stability
- Control of synchronization: pinning control, control of synchronization of a group of nodes