Namur Institute for Complex Systems

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UNIVERSITÉ DE NAMUR

Doctoral School FNRS
Nonlinear phenomena, complex systems and statistical mechanics

1-2-3 February 2022 - 9h30-12h30 – Université de Namur

Introduction to dynamical systems on complex networks

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Lecture 6 – Network-based control of multi-agent systems

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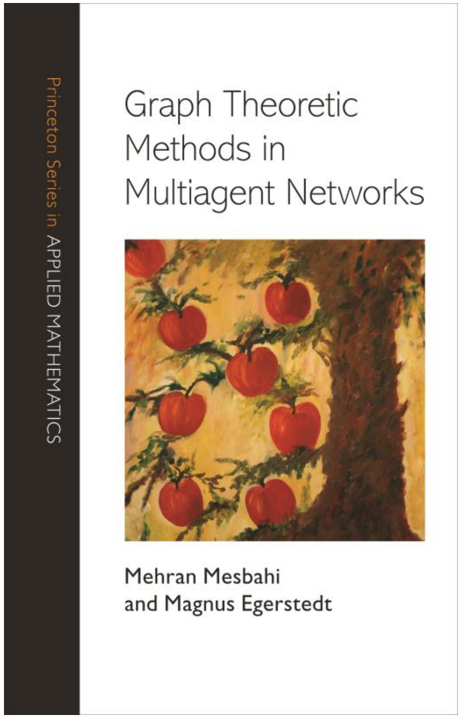
Contents

- Consensus
- The rendezvous problem
- Formation specification
- Formation control
- Control of unicycles
- Mobile robots



Consensus of integrators

$$\dot{x}_i = f(x_i) + \sigma \sum_j a_{ij} h(x_j, x_i) + u_i$$

- f=0, no pristine network (it has to be designed)

$$\Rightarrow \quad \dot{x}_i = u_i \qquad u_i = k \sum_j a_{ij}{}'(x_j - x_i)$$

$$\Rightarrow \quad \dot{x} = -Lx$$

- Control problem: to design a communication protocol such that consensus is achieved

Consensus of integrators

- Consensus dynamics

$$\dot{x} = -Lx$$

- Disagreement function

$$\varphi(x) = \frac{1}{2} x^T L x$$

- Gradient-descent algorithm

$$\dot{x} = -\nabla \varphi(x)$$

Let G be a connected undirected graph. Then, the gradient-descent algorithm asympotically solves an average-consensus problem for all initial states.

Consensus of integrators: directed graphs

- Convergence is guaranteed if the graph is strongly connected.
- Average-consensus if the network is balanced, i.e.,

$$\sum_{j \neq i} a_{ij} = \sum_{j \neq i} a_{ji}$$

R. Olfati-Saber, A. Fax, R. M. Murray, Consensus and cooperation in networked multi-agent syst*ems, Proceedings of IEEE*, 96, 2007

Consensus of heterogeneous first-order linear dynamical units

- Model

$$\dot{x}_i = \rho x_i + \delta_i + u_i$$

It's a PID!

- Consensus manifold

$$\mathfrak{C} := \left\{ \mathbf{x} \in \mathbb{R}^N \middle| \, |x_j(t) - x_i(t)| = 0, \, \forall i, j \in \mathcal{N}, i \neq j \right\}$$

- Admissible consensus

$$\lim_{t \to \infty} \mathbf{x}(t) \in \mathfrak{C}, \quad |u_i(t)| < +\infty, \quad \forall t \geq 0, \; i \in \mathcal{N}$$

- Control law

$$u_i(t) = -\sum_{j=1}^{N} \mathcal{L}_{ij} \left(\alpha x_j(t) + \beta \int_0^t x_j(\tau) d\tau + \gamma \dot{x}_j(t) \right)$$

D.A. Burbano Lombana, M. Di Bernardo, Distributed PID control for consensus of homogeneous and heterogeneous networks, *IEEE Trans. Control of Network Systems*, 2,2,2005

Conditions for convergence

Theorem IV.4: The heterogeneous group of agents (22) controlled by the distributed PID strategy (23), achieves admissible consensus for any $\beta > 0$ and $\gamma \geq 0$ if the following conditions hold:

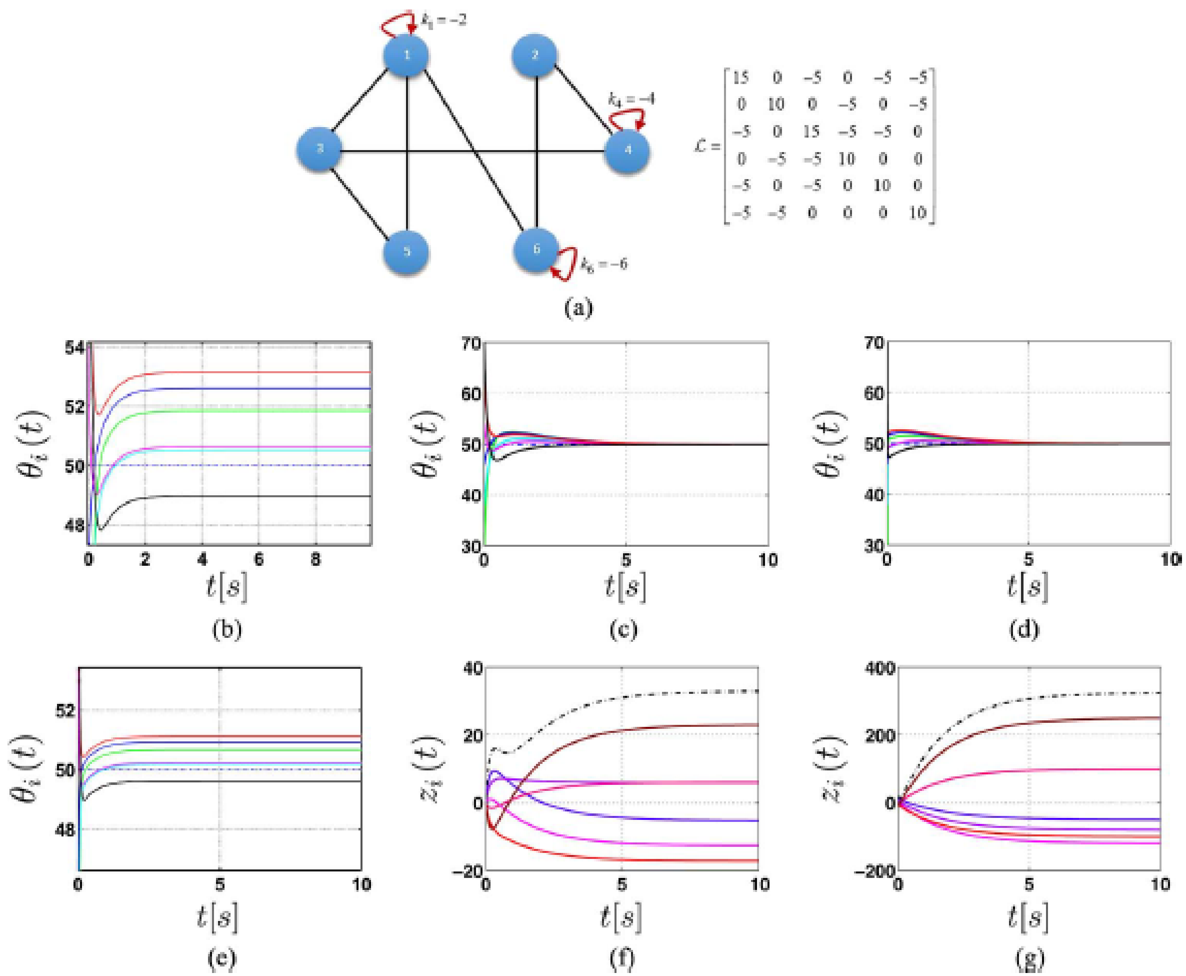
$$\psi_{11} = (1/N) \sum_{k=1}^{N} \rho_k < 0, \tag{59a}$$

$$\alpha \frac{\lambda_2}{\gamma \lambda_2 + 1} > \left(\max_i \{|\rho_i|\} + \frac{\bar{\rho}\bar{\rho}^T}{4N|\psi_{11}|} |||\mathbf{H}_1|||^2 \right) \tag{59b}$$

where $\mathbf{H}_1 := \mathbf{I}_{N-1} + \widehat{\mathbf{H}}$. Moreover, all node states converge to x_∞ as defined in Prop. III.1, and the integral actions remain bounded by z_∞ given in (44).

D.A. Burbano Lombana, M. Di Bernardo, Distributed PID control for consensus of homogeneous and heterogeneous networks, *IEEE Trans. Control of Network Systems*, 2,2,2005

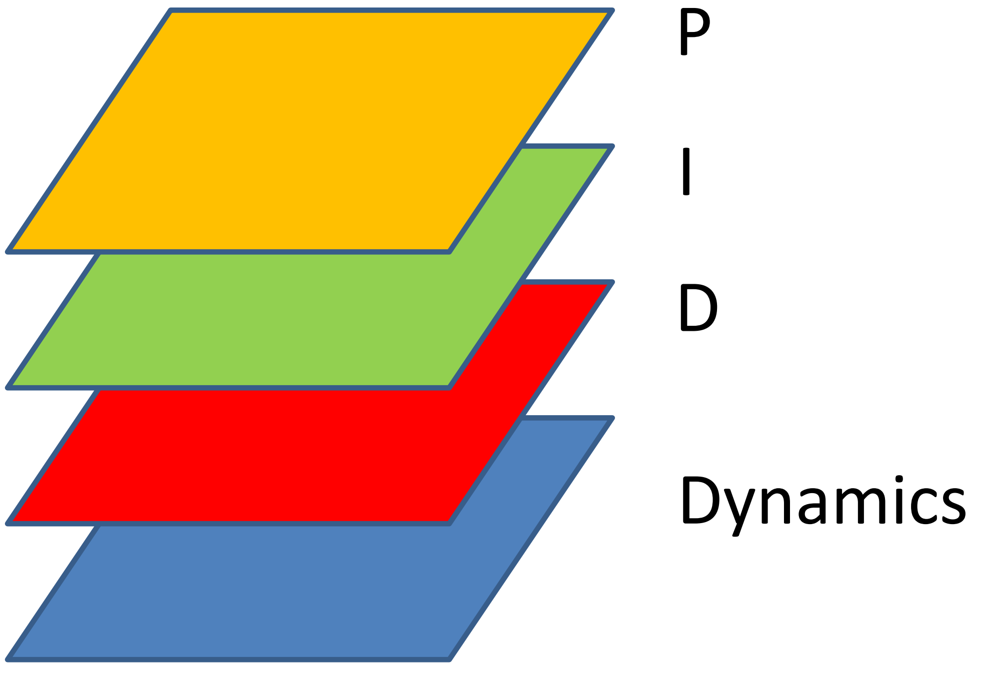
Example



P PID PI

D.A. Burbano Lombana, M. Di Bernardo, Distributed PID control for consensus of homogeneous and heterogeneous networks, *IEEE Trans. Control of Network Systems*, 2,2,2015

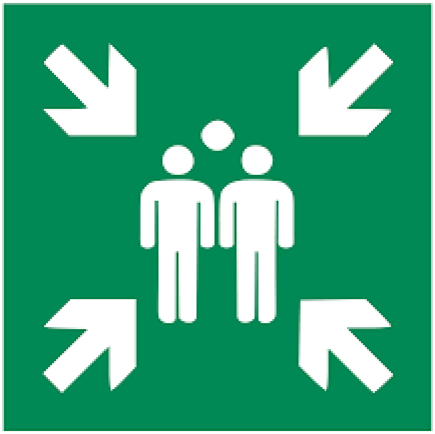
Distributed PID



- Networks do not need to be the same at each layer

The rendezvous problem

- A set of mobile agents with single integrator dynamics
- They have to agree on a single location where to meet
- Agents have not access to their global positions, but only to their relative displacements



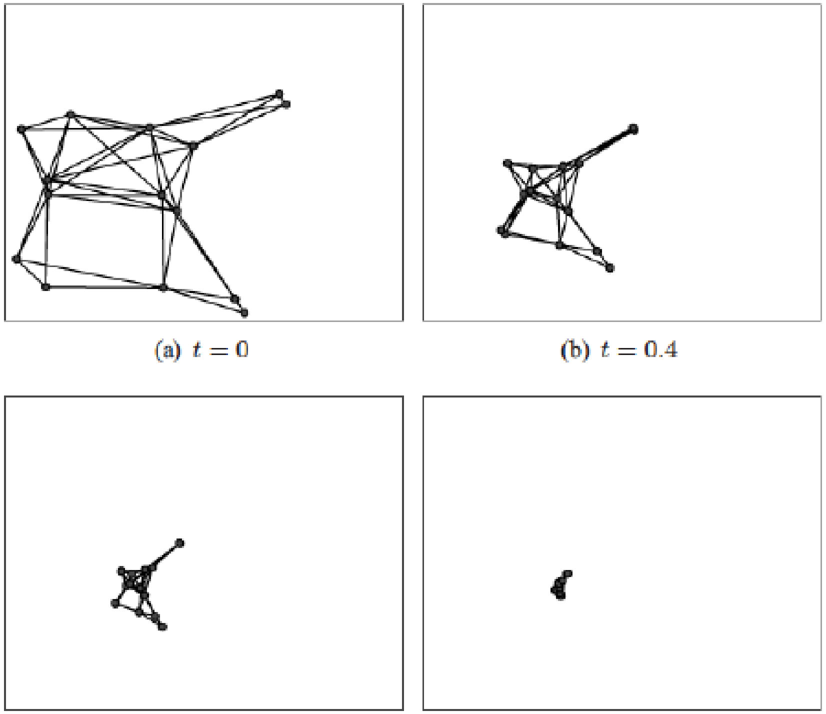
The rendezvous problem

- Agreement protocol

$$\dot{x}_i(t) = -\sum_{j \epsilon N(i)} (x_i(t) - x_j(t))$$

with $x_i(t) \epsilon R^p$

- A solution exists if the network is connected



(a) $t = 0$ (b) $t = 0.4$ (c) $t = 0.8$ (d) $t = 1.8$

Formations

- Formations are geometrical patterns realized by a multiagent team
- Agents move such that they satisfy a particular shape or relative state

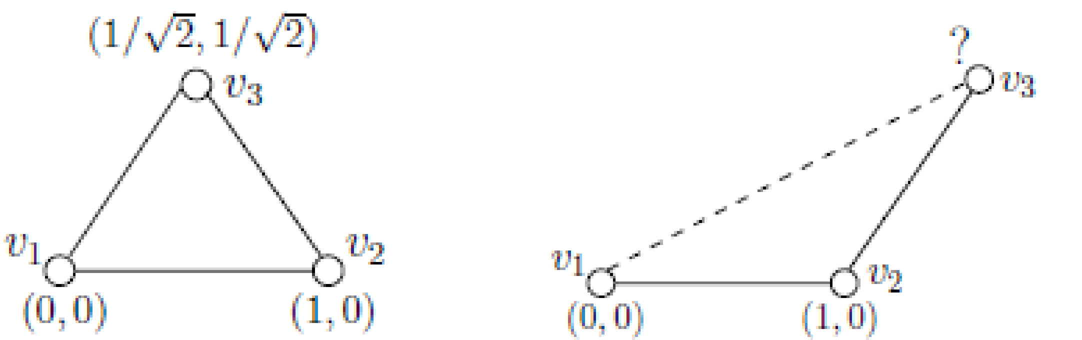
Formation shape specification

- Set of relative, desired interagent distances

$$D = \{d_{ij} \epsilon R \,|\, d_{ij} > 0, i, j = 1, \dots, N, i \neq j\}$$

- Feasible formation if there exist points $\xi_1, \dots, \xi_N$ such that

$$\|\xi_i - \xi_j\| = d_{ij} \quad for\ all\ \ i, j = 1, \dots, N, i \neq j$$



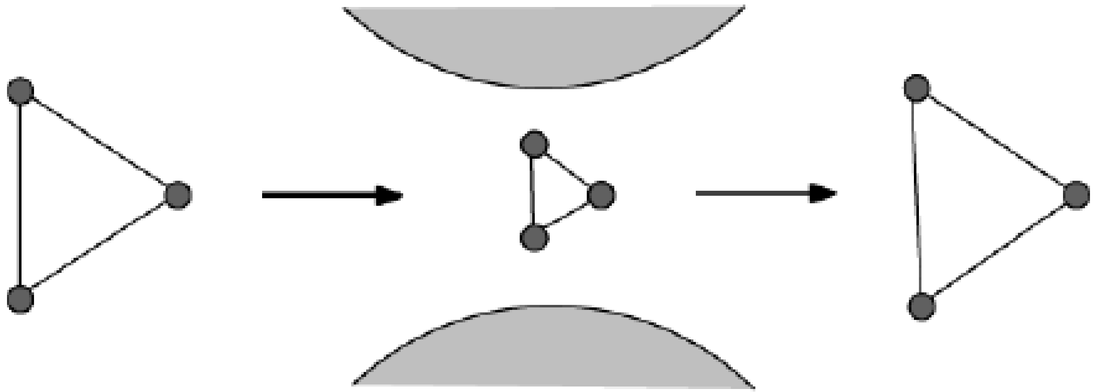
$D = \{d_{12} = d_{13} = d_{23} = 1\}$ (a) Feasible formation

$D = \{d_{12} = d_{23} = 1,\ d_{13} = 3\}$ (b) Infeasible formation

Formation shape specification

- *Scale invariant* formation

$$D' = \alpha D$$



- Application: In moderately cluttered environments, a scaled contraction or expansion of the formation may be needed to negotiate the environment

Formation shape specification

- *Translationally invariant* formation

  - Target points

$$\Xi = \{\xi_1, \dots, \xi_N\}, \quad \xi_i \epsilon R^p, i = 1, \dots, N$$

with

$$\|\xi_i - \xi_j\| = d_{ij}, i, j = 1, \dots, N, i \neq j$$

  - The points $x_1, \dots, x_N$ satisfy the formation if for some τ, then

$$x_i = \xi_i + \tau, \qquad for\ all \quad i = 1, \dots, N$$

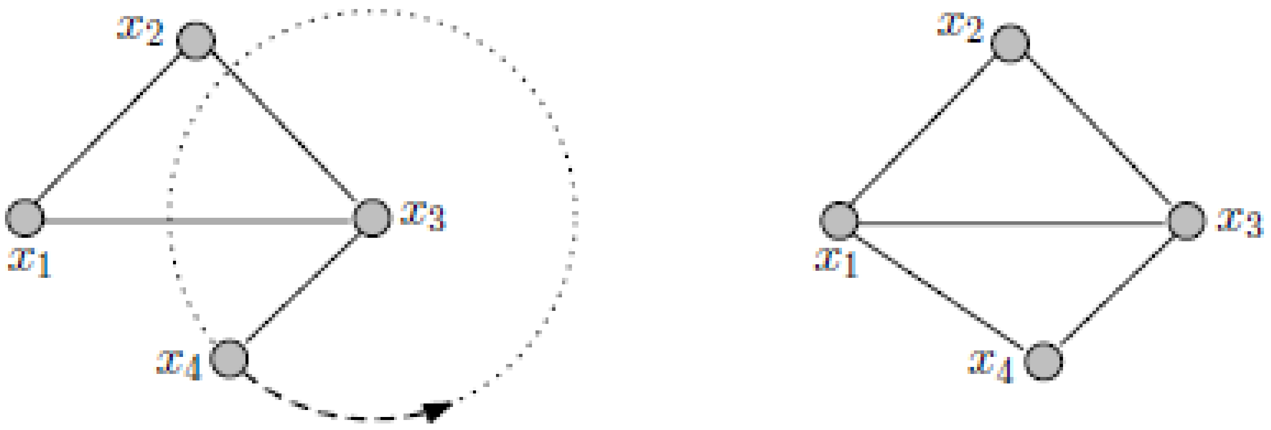
Formation shape specification

Interagent distances
$D = \{d_{ij} = d_{ji} \geq 0,\ i, j = 1, \ldots, n,\ i \neq j\}$

formation	specification	interpretation
scale invariant	D	$\lVert x_i - x_j \rVert = \alpha d_{ij}$ for some $\alpha > 0$
rigid	D	$\lVert x_i - x_j \rVert = d_{ij}$
translational invariant	Ξ	$x_i = \xi_i + \tau$ for some $\tau \in \mathbf{R}^p$

- G_f and E_f encode the edges that specify the interagent distances defined by the formation

Rigid vs. flexible formations



Formation specification through relative states

- Relative states:

$$z(t) = [(x_1(t) - x_2(t))^T, (x_2(t) - x_3(t))^T, \ldots]^T$$

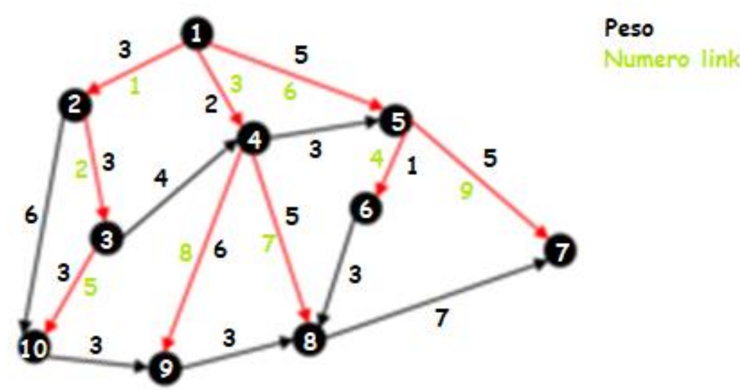
- or (with respect to an inertial frame)

$$z(t) = [(x_0 - x_1(t))^T, (x_1(t) - x_2(t))^T, \ldots]^T$$

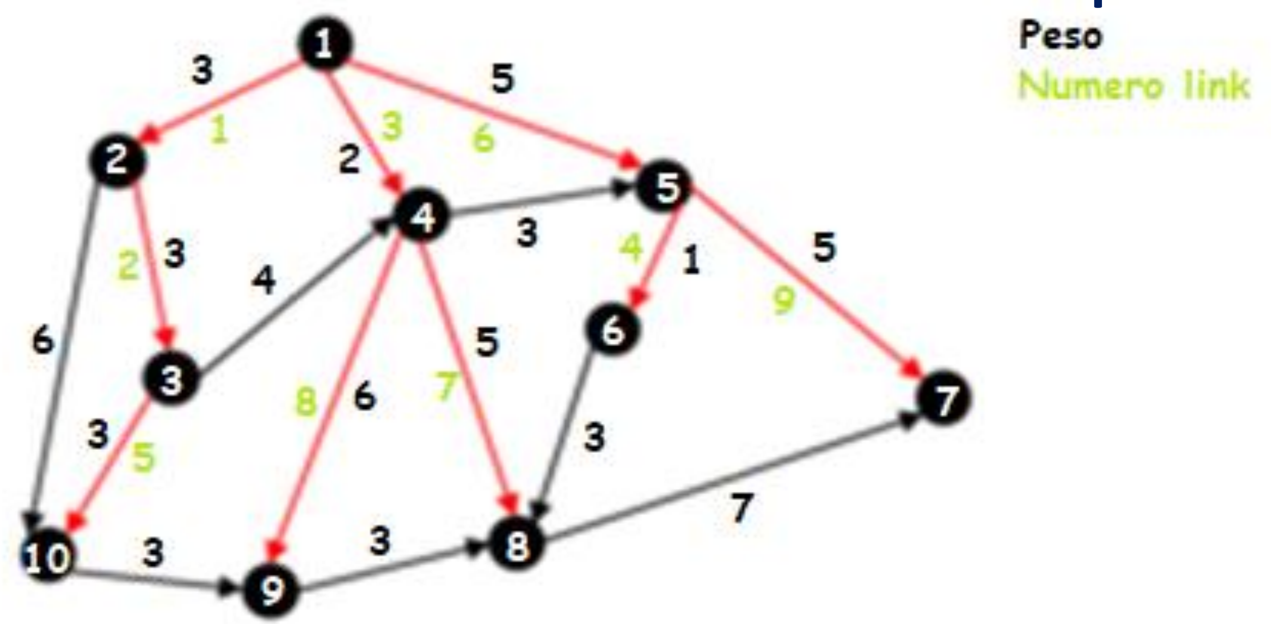
- In compact form (with B being the incidence matrix of a directed spanning tree of the graph):

$$z(t) = B^T x(t)$$

$$B = \begin{bmatrix} 1 & 0 & \\ -1 & 1 & \cdots \\ 0 & -1 & \\ & \vdots & \end{bmatrix}$$



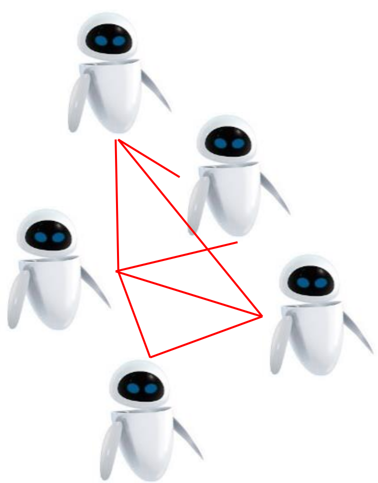
Formation specification through relative states - Example



$$B = \begin{pmatrix} 1 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & -1 & 0 & 0 & 1 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \end{pmatrix} \quad W = \begin{pmatrix} 3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 3 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 5 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 5 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 6 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 5 \end{pmatrix}$$

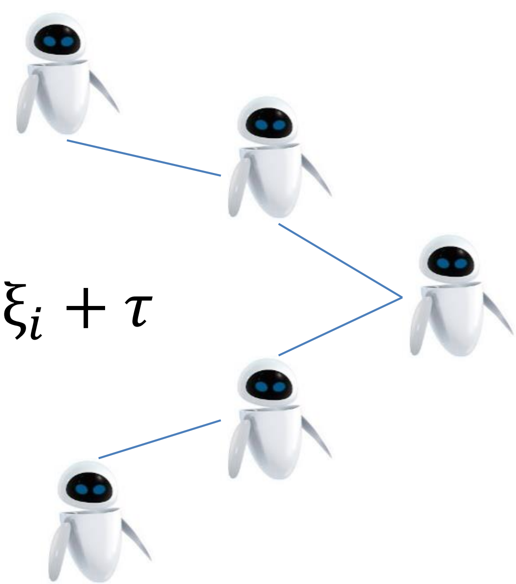
Shape-based control

- Goal: drive a set of mobile agents to a translational invariant formation specified by $G_f(V,E_f)$ and target location Ξ



Interaction graph G(V,E)

$$x_i = \xi_i + \tau$$



Target formation graph $G_f(V,E_f)$

Shape-based control: static case

- Static interaction graph G with $E_f \subseteq E$
- Define the displacement from target location

$$\tau_i(t) = x_i(t) - \xi_i$$

- Agreement protocol over displacement

$$\dot{\tau}_i(t) = -\sum_{j \epsilon N_f(i)} (\tau_i(t) - \tau_j(t))$$

- Protocol for shape-based control

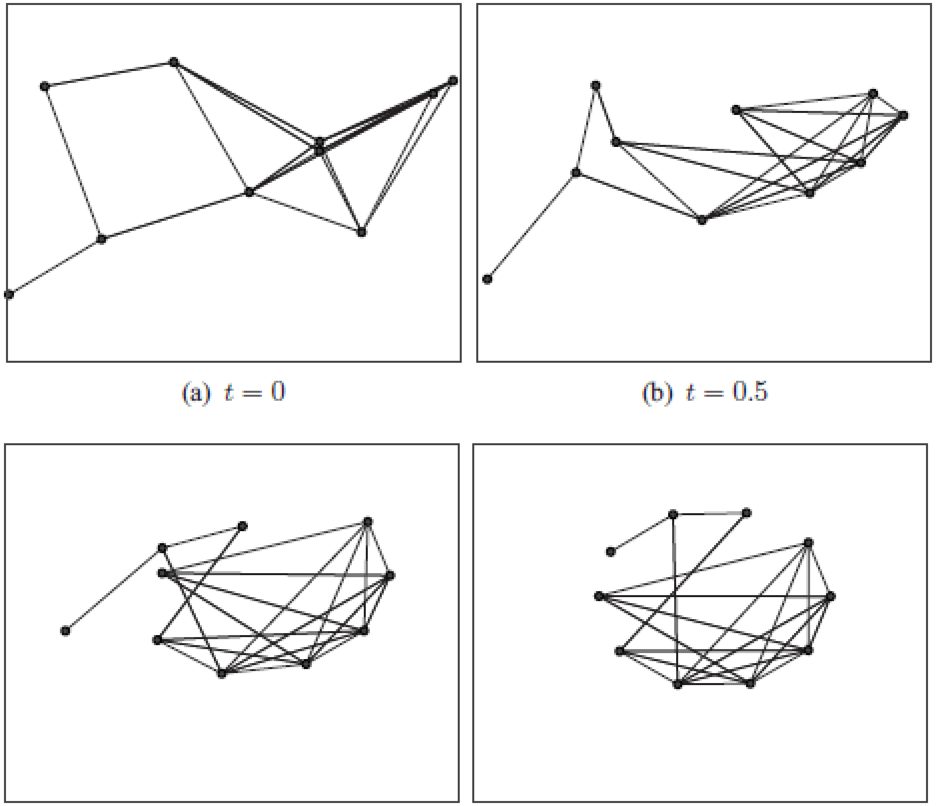
$$\dot{x}_i(t) = -\sum_{j \epsilon N_f(i)} \left[\left(x_i(t) - x_j(t)\right) - \left(\xi_i - \xi_j\right)\right] \qquad (6.7)$$

Shape-based control: static case

Theorem 6.12. *Consider the connected target formation graph $\mathcal{G}_f$ given by (V, E_f) and a set of target locations Ξ. If the static interaction graph $\mathcal{G} = (V, E)$ satisfies $E_f \subseteq E$, then the protocol (6.7) will asymptotically drive all agents to a constant displacement of the target positions, that is, for all i,*

$$x_i(t) - \xi_i \to \tau$$

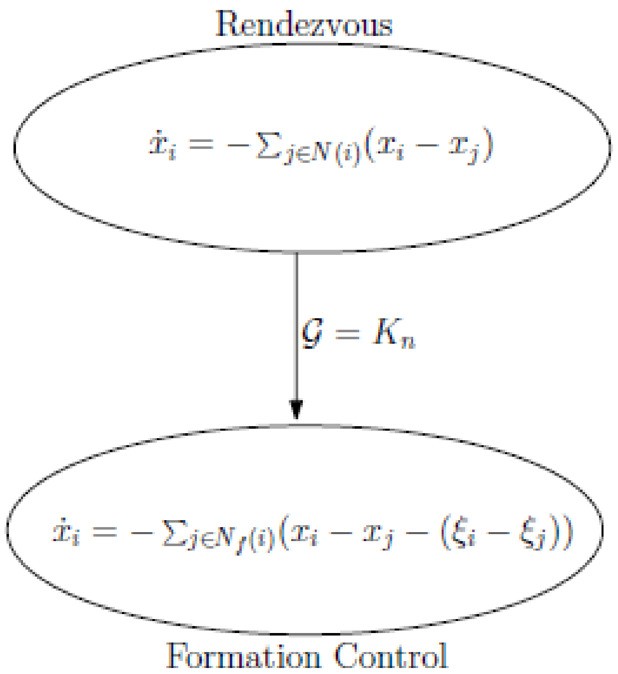
as $t \to \infty$.



(a) $t = 0$ (b) $t = 0.5$ (c) $t = 2$ (d) $t = 5$

Shape-based control: dynamic case

- The problem is solved similarly to the static case, it for all t>0, then $E_f \subseteq E(t)$
- Otherwise, if, for instance, agents interact according to a proximity graph, then one can first solve the rendezvous problem and then apply the protocol for shape control



- Enforcing $E_f \subseteq E(t)$, however, is a problem that cannot be solved by linear methods alone

Relative state-based control

- Goal: desired formation specified by a spanning tree digraph $z(t) = B^T x(t)$ and a constant reference relative position z_{ref}
- Dynamics of the agents: single integrator $\dot{x}_i(t) = u_i$
- Define the formation error as

$$e(t) = z_{ref} - z(t)$$

$$\Rightarrow \quad \dot{e}(t) = -B^T u(t)$$

- Consider the state feedback control

$$u(t) = kBe(t)$$

Relative state-based control

- Closed-loop error dynamics

$$\dot{e}(t) = -kB^T Be(t)$$

- B^TB is the edge Laplacian (Laplacian $L=BB^T$, edge Laplacian $L_e=B^TB$)
- Since the edge Laplacian of a directed spanning tree is positive definite, then

$$\lim_{t \to +\infty} e(t) = 0$$

- Closed-loop system

$$\dot{x}(t) = -kBB^T x(t) + kBz_{ref}$$

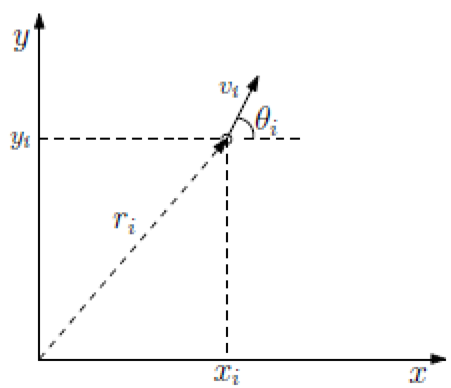
Relative state-based control

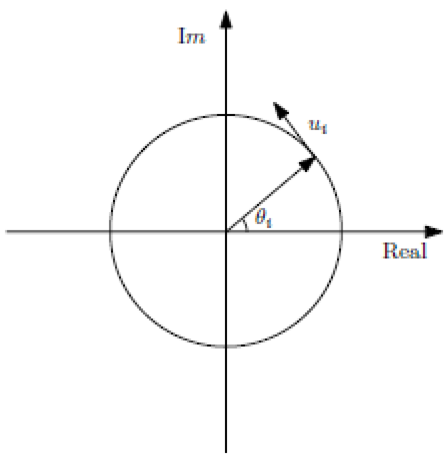
Dynamics	Control
$\dot{x}_i(t) = u_i$	$\dot{x}(t) = -kLx(t) + kBz_{ref}$
$\ddot{x}_i(t) = u_i$	$\ddot{x}(t) = -kLx(t) - kL\dot{x}(t)$ $+kBz_{ref}(t) + kB\dot{z}_{ref}(t)$
$\dot{x}_i(t) = ax_i(t) + bu_i(t)$	$\dot{z}(t) = (aI - kbL_e)z(t) + kBL_ez_{ref}$ stable if $a - \lambda_i(L)kb < 0$

Control of unicycles

- Unicycles are convenient models in aerospace (unmanned aerial vehicles) and biology (fish locomotion)
- Unicycle model

$$r_i(t) = x_i(t) + jy_i(t)$$





$$\dot{r}_i(t) = v_i e^{-j\theta_i(t)}$$
$$\dot{\theta}_i(t) = u_i(t)$$

Control of unicycles – gradient control law

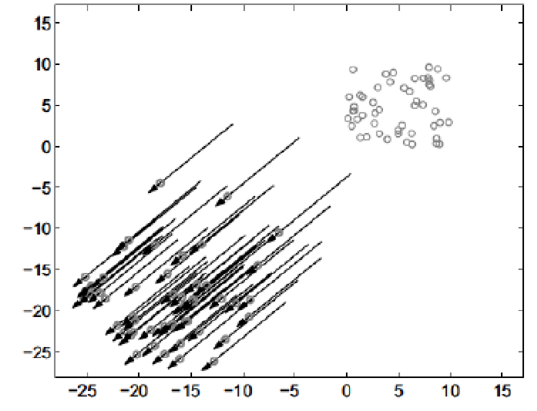
- Gradient control law

$$u_i(t) = -k\nabla_i U(\theta)$$

$$u_i(t) = -\frac{k}{N}\sum_{j=1}^{N}\sin(\theta_j(t) - \theta_i(t))$$

$$\dot{\theta}_i(t) = -\frac{k}{N}\sum_{j=1}^{N}\sin(\theta_j(t) - \theta_i(t))$$

- k>0, control of unicycle headings



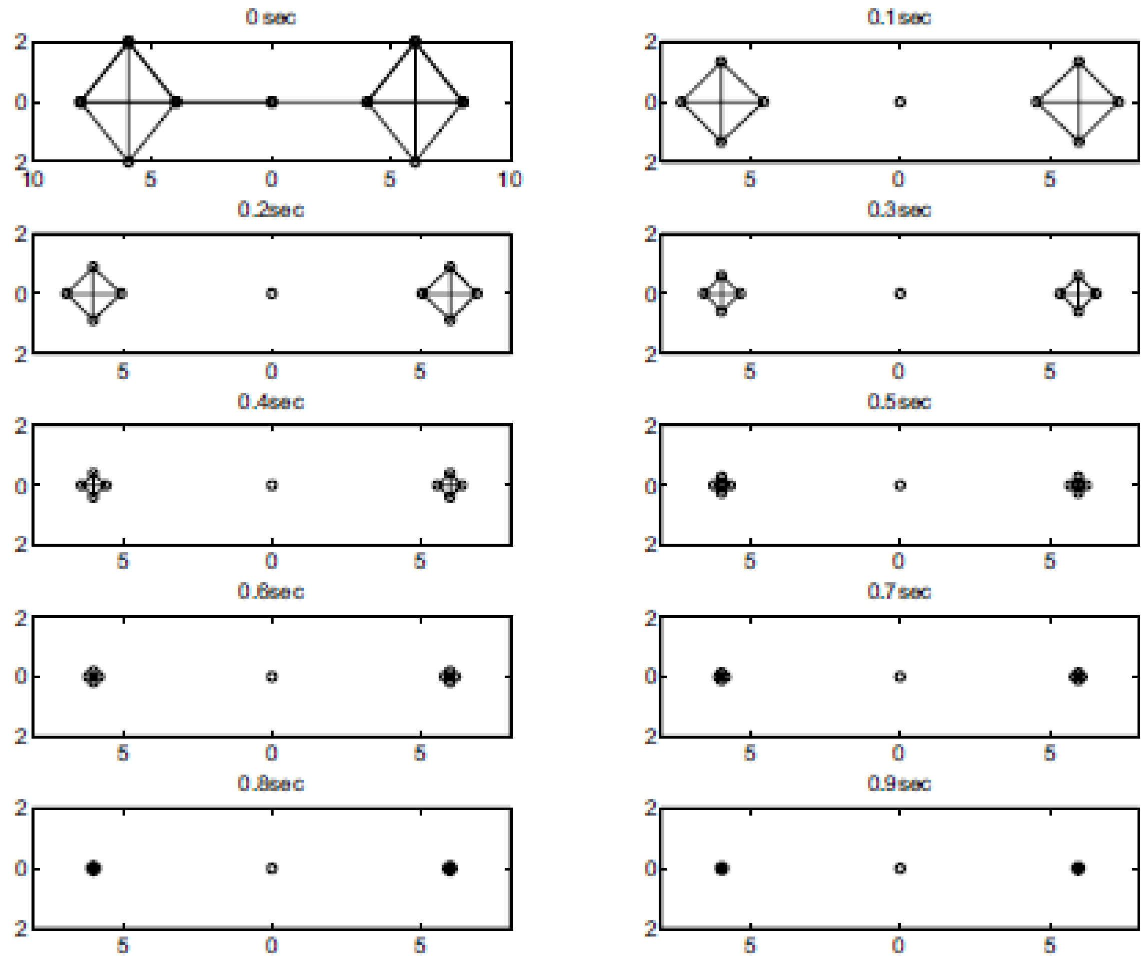
Mobile robots

- Mobile robots interacting through r-disk proximity graphs (limited sensing capabilities)

- Problem: how to keep the connectivity as robots move?

Mobile robots

- Rendezvous problem
- Linear agreement protocol
- Connectivity may be lost

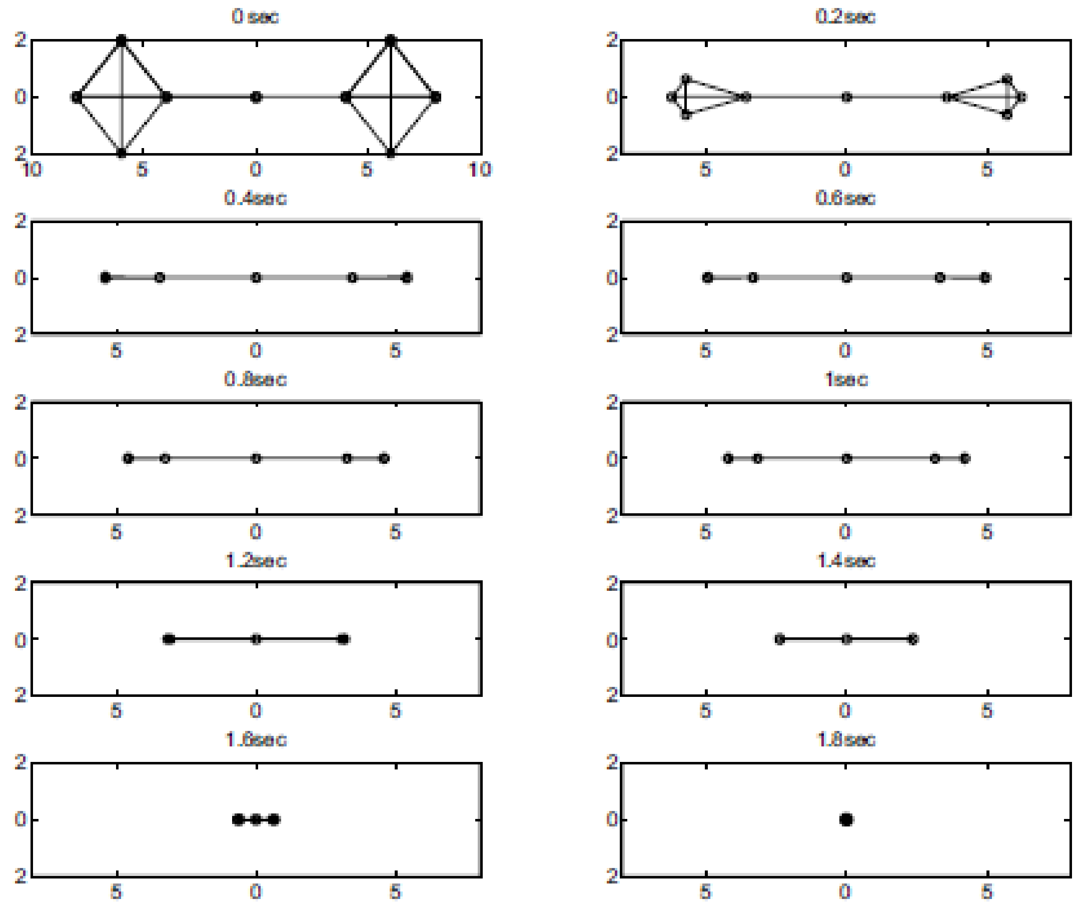


Mobile robots

- Rendezvous problem
- Nonlinear agreement protocol

$$\dot{x}_i(t) = -\sum_{j\in\mathcal{N}(i)} \frac{2\delta - \|\ell_{ij}(x(t))\|}{(\delta - \|\ell_{ij}(x(t))\|)^2}\,(x_i(t) - x_j(t)).$$

- Connectivity preserved



Summary

- Graph-based control of multiagent systems: from rendezvous problem to formation control

- Networks are the key to perform a cooperative behavior, but there is still a gap between the control technique and its usability in practice